이학박사 학위논문

Almost 2-universal quinary quadratic forms

(거의 모든 2-보편 5변수 이차형식)

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서울대학교 대학원

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Almost 2-universal quinary quadratic forms

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Abstract

In this thesis, we consider various problems on the representations of binary quadratic forms by positive definite quinary quadratic forms. A (positive definite integral) quadratic form is called *almost 2-universal* if it represents all (positive definite integral) binary quadratic forms but finitely many up to isometry. We prove that three candidates of almost 2-universal quinary diagonal quadratic forms are indeed almost 2-universal. We also prove that there are at most 16 almost 2-universal quinary quadratic forms having exactly only one exception, and prove the almost 2-universalities of some candidates. A (positive definite integral) quadratic form is called *diagonally 2-universal* if it represents all (positive definite integral) binary diagonal quadratic forms. We prove that there are at most 34 diagonally 2-universal quinary quadratic forms, and prove the diagonally 2-universalities of 29 candidates among them.

Key words: almost 2-universal, diagonally 2-universal
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Contents

Chapter 1

Introduction

A (positive definite integral) quadratic form is called n-universal if it represents all (positive definite integral) quadratic forms of rank n. In particular, if n is 1, then we simply say universal. The famous Lagrange's *four square theorem* states that the quaternary quadratic form $x^2+y^2+z^2+t^2$ is universal. After Lagrange theorem, a number of universal quadratic forms have been found by many authors (see, for example, [19] and [20]). In 2002, Conway Schneeberger proved that there are exactly 204 (classically integral) universal quaternary quadratic forms. Furthermore, they proved the so-called "15-theorem," which states that every positive definite integral quadratic form that represents

$$1, 2, 3, 5, 6, 7, 10, 14, \text{ and } 15$$

is, in fact, universal, regardless of its rank (see [1], and [2] for non-classic case).

Hereafter, we always assume that every quadratic form is *positive definite* and *classically integral*, unless stated otherwise.

In 1926, Kloosterman [11] determined all diagonal quaternary quadratic forms that represent all sufficiently large integers, which we call *almost universal forms*, although he did not succeed in proving the almost universality of four candidates. Later Pall [17] proved the almost universalities for the remaining four quadratic forms, and in fact, there are exactly 199 almost universal quaternary diagonal quadratic forms that are anisotropic over some ring of p-adic integers. Furthermore Pall and Ross [18] proved that there exist only finitely many almost universal quaternary quadratic forms that are anisotropic over some ring of p-adic integers. On the other hand, they proved that every quaternary quadratic form L such that $L_p := L \otimes \mathbb{Z}_p$ represents all p-adic integers and is isotropic over $\mathbb{Z}_p$ for all primes p is almost universal (see also Theorem 2.1 of [5]). Therefore there are infinitely many almost universal quaternary quadratic forms. Recently, Bochnak and Oh gave in [3] a very effective criterion on whether or not a quaternary quadratic form is almost universal.

As a natural generalization to higher rank case, M.-H. Kim and his collaborators proved in [8] that there are exactly eleven 2-universal quinary quadratic forms. Furthermore, they proved that any positive definite quadratic forms that represents

$$x^2+y^2,\ 2x^2+3y^2,\ 3x^2+3y^2,\ 2x^2+2xy+2y^2,\ 2x^2+2xy+3y^2,\ 2x^2+2xy+4y^2$$

is, in fact, 2-universal. For the minimal ranks of n-universal quadratic forms, see [14].

As a generalization of a result of Halmos [4] on almost universal quaternary diagonal quadratic forms with only one exception, Hwang [6] proved that there are exactly 3 quinary diagonal quadratic forms that represents all binary quadratic forms except only one. In [13], Oh proved that there exist only finitely many quinary quadratic forms that represent all but at most finitely many equivalence classes of binary quadratic forms. Such quadratic forms are called *almost 2-universal* quadratic forms. And he provided a list of almost 2-universal quinary diagonal quadratic forms, including 3 unconfirmed candidates.

In this thesis, we consider various problems on the representations of binary quadratic forms by quinary quadratic forms.

In Chapter 2, we introduce some definitions and well-known theorems on quadratic forms. We briefly summarize some well-known results on the representations of quadratic forms. We also explain our method on finding all binary quadratic forms that are represented by a quinary quadratic form that contains a quaternary quadratic subform, for example, having class number one.

In Chapter 3, we show that there are exactly 14 almost 2-universal quinary diagonal quadratic forms by proving that 3 candidates in [13] of almost 2-universal quinary diagonal quadratic forms are, in fact, almost 2-universal. We also prove that there are at most 16 almost 2-universal quinary quadratic forms having exactly only one exception, and prove the almost 2-universalities of some candidates.

In Chapter 4, we consider the quadratic forms which represents all binary diagonal quadratic forms. Such quadratic forms are called *diagonally 2-universal*. We prove that there are at most 34 diagonally 2-universal quinary quadratic forms, and prove the diagonally 2-universalities of 29 candidates among them.

Chapter 2

Preliminaries and main tools

2.1 Definitions

We adopt lattice theoretic language. Let $\mathbb{Q}$ be the rational number field. For a prime p (including ∞), let $\mathbb{Q}_p$ be the fields of p-adic completions of $\mathbb{Q}$, in particular $\mathbb{Q}_\infty = \mathbb{R}$, field of real numbers. For a finite prime p, $\mathbb{Z}_p$ denotes the p-adic integer ring. Let R be the ring of integers $\mathbb{Z}$ or the ring of p-adic integers $\mathbb{Z}_p$. An *R-lattice* L is a free R-module of finite rank equipped with a non-degenerate symmetric bilinear form $B : L \times L \to R$. The corresponding quadratic map is denoted by Q. For a R-lattice $L = R\mathbf{e}_1 + R\mathbf{e}_2 + \cdots + R\mathbf{e}_n$ with basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, we write

$$L = \big(B(\mathbf{e}_i, \mathbf{e}_j)\big).$$

For R-sublattices L_1, L_2 of L, we write $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ and $B(\mathbf{v}_1, \mathbf{v}_2) = 0$ for all $\mathbf{v}_1 \in L_1$, $\mathbf{v}_2 \in L_2$. If L admits an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, we call L *diagonal* and simply write

$$L = \big\langle Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n)\big\rangle.$$

We call L *non-diagonal* otherwise. Define the *discriminant* dL of L to be the determinant of the matrix $\big(B(\mathbf{e}_i, \mathbf{e}_j)\big)$. Note that dL is independent of the choice of a basis up to unit squares of R. We define *scale* $\mathfrak{s}L$ of L to be the ideal of R generated by $B(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in L$, norm $\mathfrak{n}L$ of L to be the ideal of R generated by $Q(\mathbf{v})$ for all $\mathbf{v} \in L$. For $a \in R^\times$, we denote by L^a the R-lattice obtained from scaling L by a.

Let ℓ, L be R-lattices. We say L *represents* ℓ if there is an injective linear map from ℓ into L that preserves the bilinear form, and write $\ell \longrightarrow L$. Such a map will be called a *representation*. A representation is called *isometry* if it is surjective. We say two R-lattices L, K are *isometric* if there is an isometry between them, and write $L \simeq K$.

For a $\mathbb{Z}$-lattice L and a prime p, we define the $\mathbb{Z}_p$-lattice $L_p := L \otimes \mathbb{Z}_p$ and call it the *localization* of L at p. The set of all $\mathbb{Z}$-lattices that are isometric to L is called the *class* of L, denoted by $\mathrm{cls}(L)$. The set of all $\mathbb{Z}$-lattices K such that $L_p \simeq K_p$ for all primes p (including ∞) is called *genus* of L, denoted by $\mathrm{gen}(L)$. The number of non-isometric classes in $\mathrm{gen}(L)$ is called the *class number* of L, denoted by $h(L)$. If a $\mathbb{Z}$-lattice ℓ is represented by one of $\mathbb{Z}$-lattices in the genus of L, then we write $\ell \longrightarrow \mathrm{gen}(L)$.

For a $\mathbb{Z}$-lattice L, we say that L is *positive definite* or simply *positive* if $Q(\mathbf{v}) > 0$ for any $\mathbf{v} \in L$, $\mathbf{v} \neq \mathbf{0}$. Let L be a positive definite $\mathbb{Z}$-lattice. L is called *n-universal* if L represents all n-ary positive definite $\mathbb{Z}$-lattices. And L is called *almost n-universal* if L represents all n-ary positive definite $\mathbb{Z}$-lattices except those in only finitely many equivalence classes. For a fixed prime p, L is called *n-universal over $\mathbb{Z}_p$* if its localization L_p represents all n-ary $\mathbb{Z}_p$-lattices. And L is called *locally n-universal* if it is n-universal over $\mathbb{Z}_p$ for all primes p.

We will denote for convenience

$$[a, b, c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Any unexplained notations and terminologies can be found in [10] or [16].

2.2 Representations of quadratic spaces and lattices

In this section, we introduce well-known theorems concerning representations of quadratic forms over $\mathbb{Q}$ and over $\mathbb{Z}_p$. The following three theorems give a complete answer for the representations of quadratic forms over $\mathbb{Q}$.

Theorem 2.2.1. *Let U and V be quadratic spaces over $\mathbb{Q}_p$. Then U is isometric to V if and only if*

$$\dim U = \dim V,\ dU = dV,\ S_pU = S_pV.$$

Proof. See 63:20 of [16]. □

Theorem 2.2.2. *Let U and V be a quadratic spaces over $\mathbb{Q}_p$ with $\nu = \dim(V) - \dim(U) \geq 0$. Then U is represented by V if and only if*

$$\begin{array}{ll} U \simeq V & \text{when } \nu = 0, \\ U \perp \langle dU \cdot dV \rangle \simeq V & \text{when } \nu = 1, \\ U \perp H \simeq V & \text{when } \nu = 2,\ dU = -dV. \end{array}$$

Here, H denotes the hyperbolic plane.

Proof. See 63:21 of [16]. □

Theorem 2.2.3 (Hasse-Minkowski Theorem)**.** *Let U and V are quadratic space over $\mathbb{Q}$. Then U is represented by V if and only if $U_p = U \otimes \mathbb{Q}_p$ is represented by $V_p = V \otimes \mathbb{Q}_p$ for all primes p (including ∞).*

Proof. See 66:3 of [16]. □

For given two $\mathbb{Z}$-lattices L and M, if $M \longrightarrow L$ then it is clear that $M_p \longrightarrow L_p$ for all primes p. But the converse is not true in general.

Let L be a $\mathbb{Z}_p$-lattice of rank n. If $(dL)\mathbb{Z}_p = (\mathfrak{s}L)^n$, then we call L a *modular lattice*. If a modular lattice L over $\mathbb{Z}_p$ is split by a sublattice of rank 1, we say that L is *proper*. Otherwise, we say that L is *improper*. Note that every modular lattice over $\mathbb{Z}_p$ is proper if p is an odd prime.

According to §91.C of [16], L has a splitting $L = L_1 \perp L_2 \perp \cdots \perp L_t$ in which each component is modular and $\mathfrak{s}L_1 \supsetneq \mathfrak{s}L_2 \supsetneq \cdots \supsetneq \mathfrak{s}L_t$. We call such a splitting a *Jordan decomposition* (or *Jordan splitting*) for L.

Definition 2.2.4. For given $\mathbb{Z}_p$-lattices l and L, let $l = \perp_{\lambda=1}^{m} l_\lambda$ and $L = \perp_{\lambda=1}^{m} L_\lambda$ be their Jordan decompositions. We define $\mathfrak{l}_i := \perp l_\mu$, where μ runs over the indices for which $\mathfrak{s}\, l_\mu \supseteq p^i\mathbb{Z}_p$. We define $\mathfrak{L}_i := \perp L_\mu$ for L similarly.

Theorem 2.2.5. *Let l and L be $\mathbb{Z}_p$-lattices, where p is an odd prime, and let $l = \perp l_\lambda$ and $L = \perp L_\lambda$ be their Jordan decompositions. Then $l \longrightarrow L$ if and only if $\mathfrak{l}_i \otimes \mathbb{Q}_p \longrightarrow \mathfrak{L}_i \otimes \mathbb{Q}_p$ for all i.*

Proof. See Theorem 1 of [15]. □

For the case when $p = 2$, we need more notations. Let $l = \perp l_\lambda$ and $L = \perp L_\lambda$ be Jordan decompositions of l and L, respectively. We define $\mathfrak{L}_{(i)} := \perp L_\mu$ where μ runs over the indices for which $\mathfrak{n}L_\mu \supseteq 2^i\mathbb{Z}_2$, and $\mathfrak{l}_{[i]} := \perp l_\mu$ where μ runs over the indices for which $\mathfrak{s}\, l_\mu \supseteq 2^i\mathbb{Z}_2$ and in addition, over the indices for which $\mathfrak{s}\, l_\mu = 2^{i+1}\mathbb{Z}_2$ with l_μ improper, if any.

We define Δ_i for L in the following way: If L has a proper 2^{i+1}-modular component, $\Delta_i := 2^{i+1}\mathbb{Z}_2$; failing this, $\Delta_i := 2^{i+2}\mathbb{Z}_2$ if L has a proper 2^{i+2}-modular component; otherwise, $\Delta_i := 0$. We define δ_i for l similarly. Let $D_i := (d\mathfrak{L}_i)\mathbb{Z}_2$ and $d_i := (d\mathfrak{l}_i)\mathbb{Z}_2$; we put $D_i = 0$ ($d_i = 0$), if $\mathfrak{L}_i = 0$ ($\mathfrak{l}_i = 0$, respectively). Note that above definitions are all independent of Jordan decompositions that define them. For any fractional ideal $\mathfrak{A} \subseteq \mathbb{Q}_2$, we write $\mathfrak{A} \longrightarrow U$ if there is an $x \in U$ such that $Q(x)\mathbb{Z}_2 = \mathfrak{A}$.

Definition 2.2.6. We say that l has a lower type than L if the followings hold for all i:

(1) $\dim \mathfrak{l}_i \leq \dim \mathfrak{L}_i$,

(2) $d_i D_i \longrightarrow 1$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(3) $\delta_i \subseteq \Delta_i + 2^{i+2}\mathbb{Z}_2$ and $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathbb{Z}_2$ if $\dim \mathfrak{l}_i = \dim \mathfrak{L}_i$,

(4) $\Delta_{i-1} \subseteq \delta_{i-1} + 2^{i+1}\mathbb{Z}_2$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_i D_i \longrightarrow 2^{i+1}$,

(5) $\delta_i \subseteq \Delta_i + 2^{i+2}\mathbb{Z}_2$ if $\dim \mathfrak{L}_i - 1 = \dim \mathfrak{l}_i > 0$ and $d_i D_i \longrightarrow 2^{i+1}$.

For two $\mathbb{Z}_2$-lattices l and L such that $l \longrightarrow L$, we denote by L/l the quadratic space satisfying $l \otimes \mathbb{Q}_2 \perp L/l \simeq L \otimes \mathbb{Q}_2$. We write $\bar{\alpha} \longrightarrow U$ if either $\alpha \longrightarrow U$ or $5\alpha \longrightarrow U$, where $\alpha \in \mathbb{Q}_2$.

Theorem 2.2.7. *Let l have a lower type than L. Then $l \longrightarrow L$ if and only if the following conditions hold for all i:*

(6) $\Delta_i \longrightarrow \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]}$,

(7) $\delta_i \longrightarrow \mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]}$,

(8) $\mathfrak{L}_{(i+2)}/\mathfrak{l}_{[i]} \simeq H$ implies $\Delta_i\delta_i \subseteq \delta_i^2$,

(9) $\bar{2^i} \longrightarrow (2^i \perp \mathfrak{L}_{(i+1)})/\mathfrak{l}_i$,

(10) $\bar{2^i} \longrightarrow (2^i \perp \mathfrak{L}_{i+1})/\mathfrak{l}_{[i]}$.

Here, H denotes the hyperbolic plane over $\mathbb{Q}_2$.

Proof. See Theorem 3 of [15]. □

Theorem 2.2.8. *For two $\mathbb{Z}$-lattices l and L, if $l_p \longrightarrow L_p$ for all primes p (including ∞), then $l \longrightarrow \mathrm{gen}(L)$.*

Proof. See 102:5 of [16]. □

2.3 Main tools

Now we provide a technique for representations of binary $\mathbb{Z}$-lattices by certain quinary $\mathbb{Z}$-lattices, which is based on the proof of the main theorems in [6] and [8]. Let $\ell = [a, b, c]$ be a binary $\mathbb{Z}$-lattice. For any integers n, s, t, we define

$$\ell^n_{s,t} := \begin{pmatrix} a - ns^2 & b - nst \\ b - nst & c - nt^2 \end{pmatrix}.$$

Let M be a $\mathbb{Z}$-lattice. It can be verified that $\ell \longrightarrow M \perp \langle n \rangle$ if and only if there exist integers s, t such that $\ell_{s,t}^n \longrightarrow M$. If the class number of the lattice M is one, we can classify all binary lattices which are represented by M using the local representation theory.

In this thesis, we only consider the case when M is quaternary. Let p be a prime such that $p \nmid 2dM$. Then $M_p \simeq \langle 1, 1, 1, dM \rangle$. If dM is a square in $\mathbb{Z}_p$, then M is 2-universal over $\mathbb{Z}_p$. If dM is not a square, then $\ell_p \longrightarrow M_p$ if and only if ℓ_p is not isometric to any sublattices of the $\mathbb{Z}_p$-lattice $\langle p, -p\Delta \rangle$, where Δ is a non-square unit in $\mathbb{Z}_p$. In particular, $\mathfrak{s}\ell \not\subseteq p\mathbb{Z}$ implies that $\ell_p \longrightarrow M_p$.

Let $\mathfrak{P}$ be the set of primes p such that $\left(\dfrac{dM}{p}\right) = -1$. We will choose s, t such that $\gcd(a - ns^2, b - nst)$ has no prime factors in $\mathfrak{P}$. Then the scale of $\ell_{s,t} = [a - ns^2, b - nst, c - nt^2]$ is not contained in $p\mathbb{Z}$ for any prime $p \in \mathfrak{P}$. Thus we may only consider the $\mathbb{Z}_p$-structure for primes $p \mid 2dM$.

Consider the case when n has a prime factor $q \in \mathfrak{P}$, which is a hard case. If $\mathfrak{s}\ell \subseteq q\mathbb{Z}$, then $\mathfrak{s}\ell_{s,t} \subseteq q\mathbb{Z}$ for all s, t. For this reason, we have to prove this case separately.

Suppose that $\ell_{s,t}^n \longrightarrow M$ over $\mathbb{Z}_p$ for all primes p. If $\ell_{s,t}$ is positive definite, then we conclude that $\ell_{s,t}^n \longrightarrow M$, and that $\ell \longrightarrow M \perp \langle n \rangle$. The following lemma says that $\ell_{s,t}^n$ is positive definite for sufficiently large a.

Lemma 2.3.1. *Let $\ell = [a, b, c]$ be a Minkowski reduced binary $\mathbb{Z}$-lattice, that is, $2|b| \leq a \leq c$. If $a > \dfrac{4}{3}n(s^2 + |st| + t^2)$, then $\ell_{s,t}^n$ is positive definite.*

Proof. Note that

$$\begin{aligned}
d\ell_{s,t}^n &= ac - b^2 - ns^2c + 2nstb - nt^2a \\
&= \frac{1}{4}ac - b^2 + \frac{3}{4}ac - n(s^2c - 2stb + t^2a) \\
&\geq 0 + \frac{3}{4}ac - n(s^2c + |st|c + t^2c) \\
&= \frac{3}{4}c\left(a - \frac{4}{3}n(s^2 + |st| + t^2)\right) \\
&> 0.
\end{aligned}$$

This completes the proof. $\square$

Chapter 3

Almost 2-universal quinary $\mathbb{Z}$-lattices

A $\mathbb{Z}$-lattice L is called *almost 2-universal* if it represents almost all binary $\mathbb{Z}$-lattices up to isometry. Note that for any quaternary $\mathbb{Z}$-lattice ℓ, there is a prime p such that ℓ_p is not 2-universal, that is, there is a binary $\mathbb{Z}_p$ -lattice that is not represented by ℓ_p. Therefore there does not exist an almost 2-universal quaternary $\mathbb{Z}$-lattice. In 2003, Oh proved in [13] that there are only finitely many almost 2-universal quinary $\mathbb{Z}$-latices. Furthermore, he proved that there are at most 14 almost 2-universal quinary diagonal $\mathbb{Z}$-lattices. Among them, he proved the almost 2-universalities of 11 lattices.

In this chapter, we prove that 3 candidates in [13] are, in fact, almost 2-universal. Furthermore, we find some almost 2-universal quinary non-diagonal $\mathbb{Z}$-lattices.

3.1 Diagonal case

First of all, we introduce the following theorem on almost 2-universal quinary diagonal $\mathbb{Z}$-lattices:

Theorem 3.1.1. *The all almost 2-universal quinary diagonal $\mathbb{Z}$-lattices and its exceptions are the followings:*

(1) 2-universal $\mathbb{Z}$-lattices
$\langle 1,1,1,1,a\rangle \quad a=1,2,3, \qquad \langle 1,1,1,2,b\rangle \quad b=2,3.$

(2) Almost 2-universal $\mathbb{Z}$-lattices and its exceptions
$\langle 1,1,1,2,4\rangle$: $\langle 3,3\rangle$, $\quad\langle 1,1,1,1,5\rangle$: $[2,1,4]$, $[4,1,4]$, $[8,1,8]$,
$\langle 1,1,1,2,5\rangle$: $\langle 3,3\rangle$, $\quad\langle 1,1,1,2,7\rangle$: $\langle 3,3\rangle$, $\langle 6,6\rangle$,
$\langle 1,1,2,2,3\rangle$: $[2,1,2]$, $\quad\langle 1,1,2,2,5\rangle$: $[2,1,2]$, $[2,1,4]$, $[4,1,4]$, $[8,1,8]$.

(3) Candidates
$\langle 1,1,1,3,7\rangle$, $\langle 1,1,2,3,5\rangle$, $\langle 1,1,2,3,8\rangle$.

Proof. See [13]. $\square$

Now we prove that these three candidates are indeed almost 2-universal.

Theorem 3.1.2. *The quinary $\mathbb{Z}$-lattice $L=\langle 1,1,1,3,7\rangle$ represents all binary lattices except following 19 binary lattices:*

$$[2,1,3],\ [4,1,4],\ \langle 1,6\rangle,\ \langle 4,6\rangle,\ [2,1,7],\ [3,1,7],\ [4,2,7],$$
$$[6,3,7],\ [7,2,7],\ [7,3,9],\ [7,1,10],\ [10,5,10],\ [7,2,15],$$
$$\langle 10,15\rangle,\ \langle 6,16\rangle,\ [7,2,22],\ [7,1,26],\ [7,3,34],\ [10,5,47].$$

Proof. Consider the quaternary sublattice $M=\langle 1,1,1,3\rangle$ of L, which has class number one. Let $\ell:=[a,b,c]$ be a binary lattice such that $0\le 2b\le a\le c,\ a,c\ne 0$. And we define the binary lattice

$$\ell_{s,t}:=[a-7s^2,b-7st,c-7t^2]$$

for each integers s,t. Note that $\ell\longrightarrow M\perp\langle 7\rangle$ if and only if $\ell_{s,t}\longrightarrow M$ for some integers s,t.

(Step 1) First, for the case when $a<30$, we verify that $\ell=[a,b,c]\longrightarrow M\perp\langle 7\rangle$. As a sample, we only consider the case when $a=10$, $b=5$. Other cases can be verified similarly. We use the fact that

$$[3,1,c]\longrightarrow M \quad \text{if} \quad c\equiv 0,1,4,5,6 \pmod 8,$$

$$\langle 3,c\rangle\longrightarrow M \quad \text{if} \quad c\equiv 1,2,3,5,6 \pmod 8.$$

For a binary lattice $\ell = [10, 5, c]$, we can easily check the followings:

- If $c \not\equiv 3, 6, 7 \pmod 8$, then $\ell_{1,2} \simeq \langle 3, c-55\rangle \longrightarrow M$. $(c > 55)$
- If $c \equiv 6 \pmod 8$, then $\ell_{1,1} \simeq [3, 1, c-8] \longrightarrow M$. $(c > 8)$
- If $c \equiv 7, 11 \pmod{16}$, then $\ell_{1,1} \subseteq [3, 1, \frac{1}{4}(c-7)] \longrightarrow M$. $(c > 7)$
- If $c \equiv 15 \pmod{16}$, then $\ell_{1,2} \subseteq \langle 3, \frac{1}{4}(c-55)\rangle \longrightarrow M$. $(c > 55)$
- If $c \equiv 3 \pmod{16}$, then $\ell_{1,3} \subseteq [3, 1, \frac{1}{4}(c-147)] \longrightarrow M$. $(c > 147)$

For a small c such that $\ell_{1,t}$ is not positive definite, we can also check it by a direct calculation. In this case, we may easily verify that three binary lattices $[10, 5, 3] \simeq [2, 1, 3], [10, 5, 10], [10, 5, 47]$ are not represented by $M \perp \langle 7\rangle$. In short,

$$[10, 5, c] \longrightarrow M \perp \langle 7\rangle \quad \text{for any } c \neq 3, 10, 47.$$

For $a < 30$, we can verify $\ell \longrightarrow L$ except followings:

$$\langle 1, 6\rangle,\ [2, 1, 3],\ \langle 4, 6\rangle,\ [4, 1, 4],\ \langle 6, 16\rangle,\ \langle 10, 15\rangle,\ [10, 5, 10],\ [10, 5, 47]$$

and

$$[7, 1, c_1]\ (c_1 = 2, 3, 10, 26),\ \ [7, 2, c_2]\ (c_2 = 4, 7, 15, 22),\ \ [7, 3, c_3]\ (c_3 = 6, 9, 34).$$

By a direct calculation, we can verify that sublattices of above exceptions with index a power of 2 are represented by L. Hereafter, we only consider $\mathbb{Z}_2$-primitive binary lattices ℓ in the sense that ℓ_2 is the only integral lattice in $\mathbb{Q}_2 \otimes \ell$ that contains ℓ_2.

(Step 2) For a binary lattice $\ell = [a, b, c]$, suppose that $a, c \geq 30$ and $0 \leq 2b \leq a \leq c$. And suppose that $\mathfrak{s}\ell \not\subseteq 7\mathbb{Z}$.

By checking the local structure of $\ell_{s,t}$ and M over $\mathbb{Z}_2, \mathbb{Z}_3$, we obtain the following properties.

(1.1) If $(a, b, c) \equiv (1, 0, 1) \pmod 2$ and $(s, t) \equiv (1, 1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.2) If $(a,b,c) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.3) If $(a,b,c) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.4) If $(a,b,c) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.5) If $a \equiv 5 \pmod 8$ or $c \equiv 5 \pmod 8$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ for any s,t.

(1.6) If $a \equiv 1 \pmod 8$, $2 \mid s$ or $c \equiv 1 \pmod 8$, $2 \mid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.7) If $4 \mid a$, $2 \nmid s$ or $4 \mid c$, $2 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(1.8) If $a \equiv 3 \pmod 4$ or $c \equiv 3 \pmod 4$, $2 \mid b$ and $2 \nmid st$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$.

(2.1) If $3 \mid ac$ and $3 \nmid st$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

(2.2) If $(a,b,c) \equiv (1,0,1),(1,0,2),(2,0,1) \pmod 3$ and $3 \nmid st$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

(2.3) If $(a,b,c) \equiv (1,2,1),(1,2,2),(2,1,2),(2,2,1) \pmod 3$ and $st \equiv 1 \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

(2.4) If $(a,b,c) \equiv (1,1,1),(1,1,2),(2,1,1),(2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

(2.5) If $(a,b,c) \equiv (2,0,2) \pmod 3$ and $st \equiv 0 \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

Under the assumption that the binary lattice ℓ is $\mathbb{Z}_2$-primitive, above conditions cover all cases. For example, the case when $a \equiv c \equiv 3 \pmod 4$ and $2 \nmid b$ is not contained in the above cases. However, ℓ is not $\mathbb{Z}_2$-primitive in this case. Note that we do not assume that ℓ is $\mathbb{Z}_3$-primitive. Regardless of $\mathbb{Z}_3$-primitivity of ℓ, all cases are contained in above.

For all cases, we may choose $s = 1, 2$ and $t \equiv i \pmod 6$ for some suitable i. Each case can be proved similarly. We only consider the case that $\ell = [a, b, c]$ satisfies the conditions given in both (1.4) and (2.3). In this case, $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2, \mathbb{Z}_3$ if $s = 2$ and $t \equiv 1 \pmod 6$.

Let $\mathfrak{P} = \{5, 7, 17, 19, 29, 31, ...\}$ be the set of primes p such that $\left(\frac{3}{p}\right) = -1$. From the assumption that $\mathfrak{s}\ell \not\subseteq 7\mathbb{Z}$, we get $\mathfrak{s}\ell_{s,t} \not\subseteq 7\mathbb{Z}$ for all s, t, and hence $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_7$. Let $p_1, p_2, ..., p_k$ be the primes in $\mathfrak{P} - \{7\}$ dividing $a - 28$. We want choose a suitable t such that $b - 14t$ is relatively prime to $p_1p_2...p_k$. Then we get $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ for all $p \neq 2, 3$.

If $k = 0$, then $\ell_{2,1} \longrightarrow M$. By Lemma 2.3.1, $\ell_{2,1}$ is positive definite if $a \geq 66$. In the case when $30 \leq a \leq 65$, one can show that $\ell_{2,1}$ is also positive definite for sufficiently large c. In the case when $a = 34$, for example, $\ell_{2,1}$ is positive definite whenever $c > 39$. The remaining cases are finitely many and we can check it by a direct calculation.

If $1 \leq k \leq 4$, then $\ell_{2,t} \longrightarrow M$ for some t such that

$$t \in \left\{6m + 1 : -\left[\frac{k+1}{2}\right] \leq m \leq \left[\frac{k}{2}\right]\right\}.$$

Note that $a \geq 28 + p_1 \cdots p_k$. In the case when $k = 3, 4, 5$, all $\ell_{2,t}$ are positive definite by Lemma 2.3.1. In case that $k = 1, 2$, however, the positiveness of $\ell_{2,t}$ is not guaranteed. There are only finitely many cases such that $\ell_{2,t}$ is not positive definite. When $\ell_{2,t}$ is not positive definite, we can check that $\ell \longrightarrow L$ by a direct calculation.

If $k = 5$, then $\ell_{2,t} \longrightarrow M$ for some $t \in \{-17, -11, ...13, 19\}$. Since $a \geq 28 + 5 \cdot 17 \cdot 19 \cdot 29 \cdot 31$, $\ell_{2,19}$ is positive definite.

If $k \geq 6$, then $\ell_{2,t} \longrightarrow M$ for some $t \in \{-3\cdot k\cdot 2^k+1, ...-5, 1, ..., 3\cdot k\cdot 2^k-5\}$. Since $a \geq 28 + 5 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 41 \cdot 43^{k-6}$, all $\ell_{2,t}$ are positive definite by Lemma 3 of [8].

(Step 3) We show that $\ell \longrightarrow L$ when $\mathfrak{s}\ell \subseteq 7\mathbb{Z}$, that is, ℓ is the form of $[7a, 7b, 7c]$. If we let $\ell' = [a, b, c]$, then $\ell = \ell'^7$. Consider the quaternary lattice $K = \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Note that K has class number one and

$$\big(K \perp \langle 21\rangle\big)^7 \longrightarrow L.$$

If $\ell' \longrightarrow K \perp \langle 21\rangle$, then $\ell'^7 = \ell \longrightarrow L$. We only consider the case when $(\ell')^7$ is $\mathbb{Z}_7$-primitive. Thus we may assume that $\ell' \simeq \langle 1, -\Delta\rangle$ over $\mathbb{Z}_7$, where Δ is a nonsquare unit in $\mathbb{Z}_7$. This is equivalent to

$$d\ell' \equiv 1, 2, 4 \pmod 7.$$

Define $\ell'_{s,t} = [a - 21s^2, b - 21st, c - 21t^2]$. From the fact that $7 \nmid d\ell'$, we have $7 \nmid d\ell'_{s,t}$ and $\ell'_{s,t} \longrightarrow K$ over $\mathbb{Z}_7$ for any s, t. The followings are the sufficient conditions such that $\ell'_{s,t} \longrightarrow K$ over $\mathbb{Z}_2$ assuming that ℓ' is $\mathbb{Z}_2$-primitive:

- $(a, b, c) \equiv (1, 0, 1) \pmod 2$ and $(s, t) \equiv (1, 1) \pmod 2$;
- $(a, b, c) \equiv (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1) \pmod 2$ and $(s, t) \equiv (0, 0) \pmod 2$;
- $(a, b, c) \equiv (1, 1, 0) \pmod 2$ and $(s, t) \equiv (1, 0) \pmod 2$;
- $(a, b, c) \equiv (0, 1, 1) \pmod 2$ and $(s, t) \equiv (0, 1) \pmod 2$.

Using the same method in Steps 1 and 2, we get $\ell' \longrightarrow K \perp \langle 21\rangle$ with finitely many exceptions. For example, $\langle 5, 61\rangle$, $[9, 2, 13] \not\longrightarrow K \perp \langle 21\rangle$. By a direct computation, one may show that the binary lattices obtained by scaling these exceptions are indeed represented by L. Therefore we conclude that $\ell \longrightarrow L$ for any binary lattice whose scale is contained in $7\mathbb{Z}$.

This completes the proof. □

For the almost 2-universalities of other $\mathbb{Z}$-lattices, the proofs are quite similar to the above. We only provide following data for the proof of almost 2-universality of L:

(1) Quternary sublattice M which has class number one,

(2) The integer n satisfying $M \perp \langle n \rangle \longrightarrow L$,

(3) Conditions such that $\ell^n_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p \mid 2dM$,

(4) Data for the case when $\mathfrak{s}\ell \subseteq q\mathbb{Z}$, where $q \mid n$ and $\left(\dfrac{dM}{q}\right) = -1$.

Theorem 3.1.3. (1) *The quinary $\mathbb{Z}$-lattice $\langle 1, 1, 2, 3, 5 \rangle$ represents all binary lattices except*

$$[2, 1, 2],\ [5, 2, 5]\ \textit{and}\ [6, 3, 6].$$

(2) *The quinary $\mathbb{Z}$-lattice $\langle 1, 1, 2, 3, 8 \rangle$ represents all binary lattices except the following 15 binary lattices:*

$$[2, 1, 2], \langle 2, 6 \rangle,\ \langle 5, 6 \rangle,\ [5, 2, 5],\ [5, 1, 8],\ [6, 3, 6],\ [6, 3, 8],\ [6, 3, 14],$$

$$\langle 6, 26 \rangle,\ \langle 7, 7 \rangle,\ [8, 4, 8],\ \langle 10, 33 \rangle,\ \langle 11, 14 \rangle,\ \langle 14, 14 \rangle,\ [25, 3, 25].$$

Proof. Let $M = \langle 1, 1, 2, 3 \rangle$ and $n = 5$ or 8. First, we collect some information on the representations over the local ring:

The followings are the conditions such that $\ell^n_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ where $n = 5, 8$ ($\mathbb{Z}_3$-primitivity of ℓ is not necessary):

- $3 \mid ac$ and $3 \nmid st$;
- $(a, b, c) \equiv (1, 0, 2), (2, 0, 1), (2, 0, 2) \pmod 3$ and $3 \nmid st$;
- $(a, b, c) \equiv (1, 1, 2), (1, 2, 1), (2, 1, 1), (2, 1, 2) \pmod 3$ and $st \equiv 1 \pmod 3$;
- $(a, b, c) \equiv (1, 1, 1), (1, 2, 2), (2, 2, 1), (2, 2, 2) \pmod 3$ and $st \equiv 2 \pmod 3$;
- $(a, b, c) \equiv (1, 0, 1) \pmod 3$ and $st \equiv 0 \pmod 3$.

The followings are the conditions such that $\ell^5_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ (ℓ is $\mathbb{Z}_2$-primitive):

- $(a, b, c) \equiv (0, 1, 0) \pmod 2$ and $\forall s, t$;
- $(a, b, c) \equiv (0, 0, 1), (1, 0, 0), (1, 0, 1) \pmod 2$ and $(s, t) \equiv (1, 1) \pmod 2$;

- $(a,b,c) \equiv (0,1,1),(1,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$;
- $(a,b,c) \equiv (1,1,0),(1,1,1) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$.

The followings are the conditions such that $\ell^8_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ (ℓ is $\mathbb{Z}_2$-primitive):

- $(a,b,c) \equiv (0,1,0),(0,1,1),(1,1,0),(1,0,1) \pmod 2$ and $\forall s,t$;
- $a \equiv 5,7 \pmod 8$ or $c \equiv 5,7 \pmod 8$ and $\forall s,t$;
- $a \equiv 10,14 \pmod{16}$, $2 \nmid s$ or $c \equiv 10,14 \pmod{16}$, $2 \nmid t$;
- $(a,c) \equiv (1,3),(9,11) \pmod{16}$, $b \equiv \pm 1 \pmod 8$ and $(s,t) \equiv (1,1) \pmod 2$;
- $(a,c) \equiv (1,11),(3,9) \pmod{16}$, $b \equiv \pm 3 \pmod 8$ and $(s,t) \equiv (1,1) \pmod 2$;
- $(a,c) \equiv (1,3),(9,11) \pmod{16}$, $b \equiv \pm 3 \pmod 8$ and $(s,t) \equiv (0,1),(1,0) \pmod 2$;
- $(a,c) \equiv (1,11),(3,9) \pmod{16}$, $b \equiv \pm 1 \pmod 8$ and $(s,t) \equiv (0,1),(1,0) \pmod 2$;
- $(a,c) \equiv (1,6),(9,6),(3,2),(11,2) \pmod{16}$, $b \equiv 2 \pmod 4$ and $(s,t) \equiv (1,0) \pmod 2$;
- $(a,c) \equiv (1,2),(9,2),(3,6),(11,6) \pmod{16}$, $b \equiv 0 \pmod 4$ and $(s,t) \equiv (1,0) \pmod 2$;
- $(a,c) \equiv (1,6),(9,6),(3,2),(11,2) \pmod{16}$, $b \equiv 0 \pmod 4$ and $(s,t) \equiv (1,1) \pmod 2$;
- $(a,c) \equiv (1,2),(9,2),(3,6),(11,6) \pmod{16}$, $b \equiv 2 \pmod 4$ and $(s,t) \equiv (1,1) \pmod 2$.

Since there are no prime factors q such that $q \mid n$ and $\left(\dfrac{dM}{q}\right) = -1$, in this case, the process such as Step 3 in the proof of Theorem 3.1.2 is not necessary. $\square$

Now we conclude the classification of almost 2-universal quinary diagonal $\mathbb{Z}$-lattices.

Theorem 3.1.4. *There are exactly 14 almost 2-universal quinary diagonal $\mathbb{Z}$-lattices. Those lattices are:*

$\langle 1,1,1,1,1\rangle$, $\langle 1,1,1,1,2\rangle$, $\langle 1,1,1,1,3\rangle$, $\langle 1,1,1,2,2\rangle$, $\langle 1,1,1,2,3\rangle$,
$\langle 1,1,1,1,5\rangle$, $\langle 1,1,1,2,4\rangle$, $\langle 1,1,1,2,5\rangle$, $\langle 1,1,1,2,7\rangle$, $\langle 1,1,2,2,3\rangle$, $\langle 1,1,2,2,5\rangle$,
$\langle 1,1,1,3,7\rangle$, $\langle 1,1,2,3,5\rangle$, $\langle 1,1,2,3,8\rangle$.

Table 3.1: Almost 2-universal quinary diagonal $\mathbb{Z}$-lattices

3.2 Non-diagonal case

In this section, we give proofs of almost 2-universalities of some quinary non-diagonal $\mathbb{Z}$-lattices.

Theorem 3.2.1. *The quinary $\mathbb{Z}$-lattice*

$$L = \langle 1,1,5\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

represents all binary lattices except $\langle 2,3\rangle$, $[5,1,5]$, $[5,2,5]$, $[5,1,11]$.

Proof. Let $M = \langle 1,1\rangle \perp [2,1,2]$ and $n = 5$.

The followings are the conditions such that $\ell^5_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$: (ℓ is $\mathbb{Z}_2$-primitive)

- $(a,b,c) \equiv (1,0,1) \pmod 2$ and $(s,t) \equiv (1,1) \pmod 2$;
- $(a,b,c) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$;
- $(a,b,c) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$;
- $(a,b,c) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$;

- $a \equiv 7 \pmod 8$ or $c \equiv 7 \pmod 8$, and $\forall s, t$;
- $a \equiv 3 \pmod 8$, $2 \mid s$ or $c \equiv 3 \pmod 8$, $2 \mid t$;
- $a \equiv 0 \pmod 4$, $2 \nmid s$ or $c \equiv 0 \pmod 4$, $2 \nmid t$;
- $a \equiv 1 \pmod 4$ or $c \equiv 1 \pmod 4$, $2 \mid b$ and $(s,t) \equiv (1,1) \pmod 2$.

The followings are the conditions such that $\ell^n_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$: ($\mathbb{Z}_3$-primitivity of ℓ is not necessary)

- $3 \mid ac$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,0,2), (2,0,1), (2,0,2) \pmod 3$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,1,2), (1,2,1), (2,1,1), (2,1,2) \pmod 3$ and $st \equiv 1 \pmod 3$;
- $(a,b,c) \equiv (1,1,1), (1,2,2), (2,2,1), (2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$;
- $(a,b,c) \equiv (1,0,1) \pmod 3$ and $st \equiv 0 \pmod 3$.

We need to consider the case when $\mathfrak{s}\ell \subseteq 5\mathbb{Z}$. Let

$$K = \langle 1, 3 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then $\big(K \perp \langle 5 \rangle\big)^5 \longrightarrow L$.
The followings are the conditions such that $\ell^5_{s,t} \longrightarrow K$ over $\mathbb{Z}_2$: (ℓ is $\mathbb{Z}_2$-primitive)

- $(a,b,c) \equiv (1,0,1) \pmod 2$ and $(s,t) \equiv (1,1) \pmod 2$;
- $(a,b,c) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0) \pmod 2$;
- $(a,b,c) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0) \pmod 2$;
- $(a,b,c) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1) \pmod 2$;
- $a \equiv 3 \pmod 8$ or $c \equiv 3 \pmod 8$, and $\forall s, t$;

- $a \equiv 7 \pmod 8$, $2 \mid s$ or $c \equiv 7 \pmod 8$, $2 \mid t$;
- $a \equiv 0 \pmod 4$, $2 \nmid s$ or $c \equiv 0 \pmod 4$, $2 \nmid t$;
- $a \equiv 1 \pmod 4$ or $c \equiv 1 \pmod 4$, $2 \mid b$ and $(s,t) \equiv (1,1) \pmod 2$.

The followings are the conditions such that $\ell^5_{s,t} \longrightarrow K$ over $\mathbb{Z}_3$: ($\mathbb{Z}_3$-primitivity of ℓ is not necessary)

- $3 \mid ac$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,0,2), (2,0,1), (2,0,2) \pmod 3$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,1,2), (1,2,1), (2,1,1), (2,1,2) \pmod 3$ and $st \equiv 1 \pmod 3$;
- $(a,b,c) \equiv (1,1,1), (1,2,2), (2,2,1), (2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$;
- $(a,b,c) \equiv (1,0,1) \pmod 3$ and $st \equiv 0 \pmod 3$. $\square$

Theorem 3.2.2. *The quinary $\mathbb{Z}$-lattice $L = \langle 1,1,3 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ represents all binary lattices except* $[2,1,2]$, $\langle 1,10 \rangle$.

Proof. Let $M = \langle 1,1,2,3 \rangle$, $n = 14$. Note that $M \perp \langle 14 \rangle \longrightarrow L$.
The followings are the conditions such that $\ell^{14}_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$: (ℓ is $\mathbb{Z}_2$-primitive)

- $(a,b,c) \equiv (0,1,0), (0,1,1), (1,0,1), (1,1,0) \pmod 2$ and $\forall s,t$;
- $a \equiv 5$ or $c \equiv 5 \pmod 8$ and $\forall s,t$;
- $a \equiv 3 \pmod 8$, $2 \nmid s$ or $c \equiv 3 \pmod 8$, $2 \nmid t$;
- $a \equiv 7 \pmod 8$, $2 \mid s$ or $c \equiv 7 \pmod 8$, $2 \mid t$;
- $a \equiv 2 \pmod 8$, $b \equiv 2 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,1) \pmod 2$;
- $a \equiv 2 \pmod 8$, $b \equiv 0 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,0) \pmod 2$;
- $a \equiv 6 \pmod 8$, $b \equiv 0 \pmod 4$ and $(s,t) \equiv (1,1) \pmod 2$;

- $a \equiv 6 \pmod 8$, $b \equiv 2 \pmod 4$ and $(s,t) \equiv (1,0) \pmod 2$.

The followings are the conditions such that $\ell_{s,t}^{14} \longrightarrow M$ over $\mathbb{Z}_3$: ($\mathbb{Z}_3$-primitivity of ℓ is not necessary)

- $3 \mid ac$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,0,2),(2,0,1),(2,0,2) \pmod 3$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,1,2),(1,2,1),(2,1,1),(2,1,2) \pmod 3$ and $st \equiv 1 \pmod 3$;
- $(a,b,c) \equiv (1,1,1),(1,2,2),(2,2,1),(2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$;
- $(a,b,c) \equiv (1,0,1) \pmod 3$ and $st \equiv 0 \pmod 3$.

For the proof that $\ell \longrightarrow L$ when $\mathfrak{s}\ell \subseteq 7\mathbb{Z}$, we consider the quaternary lattice $K = \langle 1,1,3,7\rangle$. The geus of K consists of following 3 lattices:

$$\langle 1,1,3,7\rangle,\ \langle 1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},\ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

It can be verified that $\big(K' \perp \langle 7\rangle\big)^7 \longrightarrow L$ for all $K' \in \mathrm{gen}(K)$. We denote this by

$$\big(\mathrm{gen}(K) \perp \langle 7\rangle\big)^7 \longrightarrow L.$$

Suppose that there exist integers s,t such that $\ell_{s,t}^7 \longrightarrow K$ over $\mathbb{Z}_p$ for all primes p and $\ell_{s,t}^7$ is positive definite. Then $\ell_{s,t}^7 \longrightarrow K'$ and $\ell \longrightarrow K' \perp \langle 7\rangle$ for some $K' \in \mathrm{gen}(K)$. So we get $\ell^7 \longrightarrow \big(K' \perp \langle 7\rangle\big)^7 \longrightarrow L$. Therefore, we can use the same method as step 3 in the proof of Theorem 3.1.2.

The followings are the conditions such that $\ell_{s,t}^7 \longrightarrow K$ over $\mathbb{Z}_2$:

- a is even, and s is odd;
- a is odd, and s is even.

The followings are the conditions such that $\ell_{s,t}^7 \longrightarrow K$ over $\mathbb{Z}_3$:

- $3 \mid ac$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,0,1),(1,0,2),(2,0,1) \pmod 3$ and $3 \nmid st$;
- $(a,b,c) \equiv (1,2,1),(1,2,2),(2,1,2),(2,2,1) \pmod 3$ and $st \equiv 1 \pmod 3$;
- $(a,b,c) \equiv (1,1,1),(1,1,2),(2,1,1),(2,2,2) \pmod 3$ and $st \equiv 2 \pmod 3$;
- $(a,b,c) \equiv (2,0,2) \pmod 3$ and $st \equiv 0 \pmod 3$. □

Theorem 3.2.3. *The quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

represent all binary lattices except $\langle 3,3\rangle$.

Proof. Let $M = \langle 1,1,1,2\rangle$, $n = 14, 22$. Note that

$$M \perp \langle 14\rangle \longrightarrow \langle 1,1,1\rangle \perp [2,1,4],$$

$$M \perp \langle 22\rangle \longrightarrow \langle 1,1,1\rangle \perp [2,1,6].$$

The followings are the conditions such that $\ell^n_{s,t} \longrightarrow \langle 1,1,1,2\rangle$ over $\mathbb{Z}_2$ where $n = 14, 22$: (ℓ is $\mathbb{Z}_2$-primitive)

- $(a,b,c) \equiv (0,1,0),(0,1,1),(1,0,1),(1,1,0) \pmod 2$ and $\forall s,t$;
- $a \equiv 3 \pmod 8$ or $c \equiv 3 \pmod 8$ and $\forall s,t$;
- $a \equiv 1 \pmod 8$, $2 \nmid s$ or $c \equiv 1 \pmod 8$, $2 \nmid t$;
- $a \equiv 5 \pmod 8$, $2 \mid s$ or $c \equiv 5 \pmod 8$, $2 \mid t$;
- $a \equiv 2 \pmod 8$, $b \equiv 2 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,1) \pmod 2$;
- $a \equiv 2 \pmod 8$, $b \equiv 0 \pmod 4$, $2 \nmid c$ and $(s,t) \equiv (1,0) \pmod 2$;
- $a \equiv 6 \pmod 8$, $b \equiv 0 \pmod 4$ and $(s,t) \equiv (1,1) \pmod 2$;

- $a \equiv 6 \pmod 8$, $b \equiv 2 \pmod 4$ and $(s,t) \equiv (1,0) \pmod 2$.

We need to consider the case when $L = \langle 1,1,1 \rangle \perp [2,1,6]$ and $\mathfrak{s}\ell \subseteq 11\mathbb{Z}$. Let

$$K = \langle 1,1,1,11 \rangle.$$

Then $\big(\mathrm{gen}(K) \perp \langle 11 \rangle\big)^{11} \longrightarrow \langle 1,1,2 \rangle \perp [2,1,6]$.

The followings are the conditions such that $\ell_{s,t}^{11} \longrightarrow K$ over $\mathbb{Z}_2$: (ℓ is $\mathbb{Z}_2$-primitive)

- $(a,b,c) \equiv (1,0,1) \pmod 2$ and $(s,t) \equiv (1,1)$;
- $(a,b,c) \equiv (0,1,0) \pmod 2$ and $(s,t) \equiv (0,0)$;
- $(a,b,c) \equiv (1,1,0) \pmod 2$ and $(s,t) \equiv (1,0)$;
- $(a,b,c) \equiv (0,1,1) \pmod 2$ and $(s,t) \equiv (0,1)$;
- $a \equiv 1 \pmod 8$ or $c \equiv 1 \pmod 8$, and $\forall s,t$;
- $a \equiv 5 \pmod 8$, $2 \mid s$ or $c \equiv 5 \pmod 8$, $2 \mid t$;
- $a \equiv 0 \pmod 4$, $2 \nmid s$ or $c \equiv 0 \pmod 4$, $2 \nmid t$;
- $a \equiv 3 \pmod 4$ or $c \equiv 3 \pmod 4$, $2 \mid b$ and $(s,t) \equiv (1,1) \pmod 2$. $\square$

Theorem 3.2.4. *The quinary $\mathbb{Z}$-lattice*

$$\langle 1,1,2 \rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

represents all binary lattices except $[2,1,2]$.

Proof. Let $M = \langle 1,2 \rangle \perp [3,1,3]$, $n = 1$. Since the discriminant $dM = 2^4$, M is 2-universal over $\mathbb{Z}_p$ for all odd primes p. Thus, for a positive definite binary $\mathbb{Z}$-lattice ℓ, $\ell \longrightarrow M$ over $\mathbb{Z}$ if and only if $\ell \longrightarrow M$ over $\mathbb{Z}_2$. Using this, we can verify the followings, where ℓ is $\mathbb{Z}_2$-primitive:

- If $\mathrm{ord}_2\, d\ell = 1$, then $\ell \longrightarrow M$;

- If $(a, b, c) = (0, 1, 0) \pmod 2$, then $\ell_{1,1} \longrightarrow M$ or $\ell_{1,-1} \longrightarrow M$;
- If $(a, b, c) = (0, 1, 1), (1, 0, 1), (1, 1, 0) \pmod 2$, then $\ell \longrightarrow M$, $\ell_{0,2} \longrightarrow M$ or $\ell_{2,0} \longrightarrow M$.

For $a < 5$, we can easily check that $\ell = [a, b, c] \longrightarrow L$ except $[2, 1, 2]$. The lattice $\langle 2, 6\rangle$ is the unique sublattice with index 2 of $[2, 1, 2]$ up to isometric. And $\langle 2, 6\rangle \longrightarrow L$. Thus the lattice $[2, 1, 2]$ is the only exception. $\square$

Remark 3.2.5. There are two definitions of integral lattices. A *classic* integral lattice is a lattice whose scale is contained in $\mathbb{Z}$, while a *non-classic* integral lattice is a lattice whose norm is contained in $\mathbb{Z}$. A $\mathbb{Z}$-lattice L is called *even* if $Q(\mathbf{v}) \in 2\mathbb{Z}$ for all $\mathbf{v} \in L$, and L is called *odd* otherwise. It is convenient to consider even $\mathbb{Z}$-lattices obtained from non-classic integral lattices by scaling 2. We say that an even $\mathbb{Z}$-lattice is called *even n-universal* if it represents all even $\mathbb{Z}$-lattices of rank n. The n-universal non-classic integral lattices correspond to the even n-universal classic integral lattices. For any $\mathbb{Z}$-lattice, we define

$$L^e := \{\mathbf{v} \in L \mid Q(\mathbf{v}) \equiv 0 \mod 2\}.$$

It can be verified that L^e is even 2-universal if L represents all even binary lattices. Applying this to the $\mathbb{Z}$-lattices (see Theorem 3.2.1 and 3.2.3)

$$\langle 1, 1, 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix} \text{ and } \langle 1, 1, 5\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

we obtain two even 2-universal quinary $\mathbb{Z}$-lattices. These are the candidates for even 2-universal quinary $\mathbb{Z}$-lattices provided in [12].

Corollary 3.2.6. *The quinary $\mathbb{Z}$-lattices*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$$

are even 2-universal.

3.3 Almost 2-universal quinaries having only one exception

In this section, we find all candidates for almost 2-universal quinary $\mathbb{Z}$-lattices having only one exception. For this, we need following theorem which is a criterion for 2-universality.

Theorem 3.3.1. *A $\mathbb{Z}$-lattice is 2-universal if and only if it represents the following six binary lattices:*

$$\langle 1,1\rangle,\ \langle 2,3\rangle,\ \langle 3,3\rangle,\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$

Proof. See [8]. □

Let L be an almost 2-universal $\mathbb{Z}$-lattice which is not 2-universal. Then, by above theorem, at least one of six binary lattices in the theorem is an exception of L.

Lemma 3.3.2. *Let L be a $\mathbb{Z}$-lattice.*
(1) If L represents two lattices $\langle 1,1,1,1\rangle$ and $[2,1,4]$, then L is 2-universal.
(2) If L represents two lattices $\langle 1,1,1,2\rangle$ and $\langle 3,3\rangle$, then L is 2-universal.
(3) If L represents two lattices $\langle 1,1\rangle \perp [2,1,2]$ and $\langle 2,3\rangle$, then L is 2-universal.

Proof. (1) If $a \geq 4$, then $[2,1,4] \not\to \langle 1,1,1,1,a\rangle$. Hence, L has a sublattice isometric to one of the followings:

$$\langle 1,1,1,1,1\rangle,\ \langle 1,1,1,1,2\rangle,\ \langle 1,1,1,1,3\rangle.$$

And these quinary $\mathbb{Z}$-lattices are all 2-universal.
(2) Since $\langle 3,3\rangle \not\to \langle 1,1,1,2\rangle$, L has a sublattice isometric to one of the followings:

$$\langle 1,1,1,1,2\rangle, \langle 1,1,1,2,2\rangle, \langle 1,1,1,2,3\rangle,$$

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

And these quinary $\mathbb{Z}$-lattices are all 2-universal.
(3) Since $\langle 2,3\rangle \not\longrightarrow \langle 1,1\rangle \perp [2,1,2]$, L has a sublattice isometric to one of the followings:

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

And these quinary $\mathbb{Z}$-lattices are all 2-universal. □

Theorem 3.3.3. *There are exactly four almost 2-universal quinary $\mathbb{Z}$-lattices having only one exception $\langle 3,3\rangle$, and they are*

$$\langle 1,1,1,2,4\rangle,\ \langle 1,1,1,2,5\rangle,\ \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},\ \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}.$$

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattices except $\langle 3,3\rangle$. If $\langle 1,1,1\rangle \not\longrightarrow L$, then L has a sublattice isometric to $\langle 1,1\rangle \perp [2,1,2]$ since L represents $\langle 1,1\rangle$ and $[2,1,2]$. In this case, L is 2-universal by Lemma 3.3.2. Now suppose that $\langle 1,1,1\rangle \longrightarrow L$. Then $L \simeq \langle 1,1,1\rangle \perp L_0$ for some lattice L_0. If $\mu(L_0) \geq 3$, then $[2,1,3] \not\longrightarrow L$. If $\langle 1,1,1,1\rangle \longrightarrow L$, then L is 2-universal by Lemma 3.3.2. Hence we may assume that $\langle 1,1,1,1\rangle \not\longrightarrow L$ and $\langle 1,1,1,2\rangle \longrightarrow L$. Since $\langle 6,6\rangle \not\longrightarrow \langle 1,1,1,2\rangle$, L is isometric to one of the followings:

$$\langle 1,1,1,2,4\rangle,\ \langle 1,1,1,2,5\rangle,\ \langle 1,1,1,2,6\rangle,$$

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},\ \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}.$$

Note that $[6,3,21] \not\longrightarrow \langle 1,1,1,2,6\rangle$. And the remaining four lattices represent all binary lattices except $\langle 3,3\rangle$ by Theorem 3.1.1 and Theorem 3.2.3. □

Remark 3.3.4. Note that $\langle 2,7\rangle \not\longrightarrow \langle 1,1,1,2\rangle$. Using this, it can be verified that the quinary $\mathbb{Z}$-lattice $\langle 1,1,1,2,7\rangle$ is the unique candidate for almost 2-

universal quinary $\mathbb{Z}$-lattice having only two exceptions $\langle 3,3\rangle$ and $\langle 6,6\rangle$. Indeed, $\langle 1,1,1,2,7\rangle$ represents all binary lattices except $\langle 3,3\rangle, \langle 6,6\rangle$ by Theorem 3.1.1.

Theorem 3.3.5. *There are no almost 2-universal quinary $\mathbb{Z}$-lattices having only one exception* $[2,1,4]$.

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattices except $[2,1,4]$. We can assume that L represents $\langle 1,1,1,1\rangle$ by the proof of Theorem 3.3.3 and Lemma 3.3.2. Hence $L \simeq \langle 1,1,1,1,a\rangle$ for some a. If $a \geq 4$, then $[4,1,4] \not\longrightarrow L$. And if $a \leq 3$, then L is 2-universal by Theorem 3.1.1. This completes the proof. $\square$

Remark 3.3.6. Note that there exists an almost 2-universal $\mathbb{Z}$-lattice with rank greater than 5 which have only one exception $[2,1,4]$. Consider the senary $\mathbb{Z}$-lattice

$$L = \langle 1,1,1,1\rangle \perp \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Using the fact that L has sublattices isometric to $\langle 1,1,1,1,4\rangle$ and $\langle 1,1,1,1,6\rangle$, one can show that L represents all binary lattices except $[2,1,4]$.

Theorem 3.3.7. *There are at most one almost 2-universal quinary $\mathbb{Z}$-lattice having only one exception $\langle 2,3\rangle$. The candidate is*

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattices except $\langle 2,3\rangle$. Since $\langle 2,3\rangle \longrightarrow \langle 1,1,1\rangle$, L does not represent $\langle 1,1,1\rangle$. Since L represents $\langle 1,1\rangle$ and $[2,1,2]$, L has a sublattice isometric to $\langle 1,1\rangle \perp [2,1,2]$. Since $\langle 1,5\rangle \not\longrightarrow \langle 1,1\rangle \perp [2,1,2]$, L is isometric to one of the followings:

$$\langle 1,1,4\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \ \langle 1,1,5\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

From the fact that

$$\langle 3,18\rangle \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1,4\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ [5,1,5] \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1,5\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$[5,1,5] \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix},$$

only one candidate survives. $\square$

Theorem 3.3.8. *There are at most one almost 2-universal quinary $\mathbb{Z}$-lattice having only one exception* $[2,1,3]$*. The candidate is*

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}.$$

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattices except $[2,1,3]$. We may assume that $L = \langle 1,1,1\rangle \perp L_0$ for some lattice L_0 by the proof of Theorem 3.3.3. Since $\langle 3,3\rangle \longrightarrow L$, L_0 represents 3. Since the lattices $\langle 1,1,1,1\rangle, \langle 1,1,1,2\rangle$ represent $[2,1,3]$, we get $\mu(L_0) = 3$. Since $[4,1,4] \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1,1,3\rangle$, L is isometric to one of the followings:

$$\langle 1,1,1,3,3\rangle,\ \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}.$$

From the fact that

$$[7,2,7] \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1,1,3,3\rangle,\ [2,1,7] \longrightarrow\!\!\!\!\!\!\not\quad \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

only one candidate survives. $\square$

Theorem 3.3.9. *There are at most three almost 2-universal quinary $\mathbb{Z}$-*

lattices having only one exception $\langle 1,1\rangle$. The candidates are

$$\langle 1,2\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattice except $\langle 1,1\rangle$. We may assume that $L = \langle 1\rangle \perp L_0$ for some quaternary lattice L_0 such that $\mu(L_0) = 2$. Since $\langle 2,2\rangle, [2,1,2] \longrightarrow L$, L_0 has a sublattice isometric to

$$\langle 2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Since L represents $\langle 1,3\rangle$, L_0 represents 3. Hence L is isometric to one of the followings:

$$\langle 1,2,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \langle 1,2\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},$$

$$\langle 1,3\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \ \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

$$\langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}, \ \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Deleting four lattices

$$[4,1,10] \not\longrightarrow \langle 1,2,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \ \langle 2,5\rangle \not\longrightarrow \langle 1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},$$

$$[2,1,4] \not\longrightarrow \langle 1,3\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix},\ [2,1,5] \not\longrightarrow \langle 1\rangle \perp \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix},$$

we get three candidates. □

Theorem 3.3.10. *There are two almost 2-universal quinary $\mathbb{Z}$-lattices having only one exception* $[2,1,2]$ *found so far. They are:*

$$\langle 1,1,2,2,3\rangle,\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

And there can be at most five more. The candidates are:

$$\langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix},$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}.$$

Proof. Let L be a quinary $\mathbb{Z}$-lattice representing all binary lattices except $[2,1,2]$. Since $[2,1,2] \longrightarrow \langle 1,1,1\rangle$, L does not represent $\langle 1,1,1\rangle$. Since $\langle 1,1\rangle, \langle 1,2\rangle \longrightarrow L$, L represents $\langle 1,1,2\rangle$. Since $\langle 2,3\rangle \not\longrightarrow \langle 1,1,2\rangle$, L has a sublattice isometric to one of the followings:

$$\langle 1,1,2,2\rangle,\ \langle 1,1,2,3\rangle,\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

From the fact that

$$[2,1,4] \not\longrightarrow \langle 1,1,2,2\rangle,\ [5,2,5] \not\longrightarrow \langle 1,1,2,3\rangle,\ \langle 3,3\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

L is isometric to one of the followings:

$$\langle 1,1,2,2,3\rangle,\ \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix},$$

$$\langle 1,1,2,3,3\rangle,\ \langle 1,1,2,3,4\rangle,\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix},$$

$$\langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}.$$

From the following observation, Theorem 3.1.1 and Theorem 3.2.4, we get the desired result.

$$[2,1,6] \not\to \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix},\ \langle 1,7\rangle \not\to \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix},$$

$$[2,1,18] \not\to \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},$$

$$[2,1,14] \not\to \langle 1,1,2,3,3\rangle,\ [6,3,6] \not\to \langle 1,1,2,3,4\rangle,$$

$$[6,3,6] \not\to \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,10\rangle \not\to \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}.$$ □

To sum up, we get the following 16 candidates for almost 2-universal quinary $\mathbb{Z}$-lattices having only one exception. The lattices marked bold face are indeed almost 2-universal. Their almost 2-universalities are proved in this thesis or other papers.

exception $\langle 1,1\rangle$
$\langle 1,2\rangle \perp \begin{pmatrix}2&1&0\\1&2&1\\0&1&3\end{pmatrix}$, $\quad\langle 1\rangle \perp \begin{pmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&3\end{pmatrix}$, $\quad\langle 1\rangle \perp \begin{pmatrix}2&0&0&1\\0&2&1&0\\0&1&2&1\\1&0&1&3\end{pmatrix}$,

exception $\langle 2,3\rangle$
$\langle 1,1\rangle \perp \begin{pmatrix}2&1&0\\1&2&1\\0&1&4\end{pmatrix}$,

exception $\langle 3,3\rangle$
$\langle \mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{4}\rangle$, $\quad\langle \mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{5}\rangle$,
$\langle \mathbf{1},\mathbf{1},\mathbf{1}\rangle \perp \begin{pmatrix}\mathbf{2}&\mathbf{1}\\\mathbf{1}&\mathbf{4}\end{pmatrix}$, $\quad\langle \mathbf{1},\mathbf{1},\mathbf{1}\rangle \perp \begin{pmatrix}\mathbf{2}&\mathbf{1}\\\mathbf{1}&\mathbf{6}\end{pmatrix}$,

exception $[2,1,2]$
$\langle \mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3}\rangle$, $\quad\langle 1,1,2\rangle \perp \begin{pmatrix}2&1\\1&3\end{pmatrix}$,
$\langle \mathbf{1},\mathbf{1},\mathbf{2}\rangle \perp \begin{pmatrix}\mathbf{3}&\mathbf{1}\\\mathbf{1}&\mathbf{3}\end{pmatrix}$, $\quad\langle 1,1,2\rangle \perp \begin{pmatrix}3&1\\1&4\end{pmatrix}$,
$\langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&3\end{pmatrix}$, $\quad\langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&4\end{pmatrix}$, $\quad\langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&5\end{pmatrix}$,

exception $[2,1,3]$
$\langle 1,1,1\rangle \perp \begin{pmatrix}3&1\\1&4\end{pmatrix}$.

Table 3.2: Almost 2-universal quinaries having only one exception

Chapter 4

Diagonally 2-universal $\mathbb{Z}$-lattices

In this chapter, we consider the $\mathbb{Z}$-lattices which represent all binary diagonal $\mathbb{Z}$-lattices. Such $\mathbb{Z}$-lattices are called *diagonally 2-universal* $\mathbb{Z}$-lattices. We prove that there are at most 34 diagonally 2-universal quinary $\mathbb{Z}$-lattices, and prove the diagonally 2-universalities of 29 candidates among them.

4.1 Candidates of diagonally 2-universal quinary $\mathbb{Z}$-lattices

Now we find all candidates for diagonally 2-universal quinary $\mathbb{Z}$-lattices using the escalation method. The following theorem gives the answer. In addition, it will be used to find the criterion for diagonally 2-universality applying to not only quinary $\mathbb{Z}$-lattices but also $\mathbb{Z}$-lattices with rank greater than 5.

Theorem 4.1.1. *Let L be a diagonally 2-universal $\mathbb{Z}$-lattice. Then L has a quinary sublattice which is isometric to one of the followings:*

$$\langle 1,1,1,1,a\rangle \ : \ a = 1,2,3,4,5,$$

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & a \\ a & b \end{pmatrix} \ : \ a = 0,1, \ b = 2,3,$$

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 3 & a \\ a & b \end{pmatrix} \ : \ a=0,1,\ b=3,4,5,6,$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & a \\ 0 & 2 & b \\ a & b & c \end{pmatrix} \ : \ a=0,1,\ b=0,1,\ c=2,3,4,5,$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & a \\ 0 & 3 & b \\ a & b & c \end{pmatrix} \ : \ a=0,1,\ b=0,1,\ c=3,4,5,6,$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle 3\rangle,\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Proof. Since L represents the binary lattice $\langle 1,1\rangle$, we have $L \simeq \langle 1,1\rangle \perp L_0$ for some $\mathbb{Z}$-lattice L_0. Since $\langle 1,2\rangle \not\longrightarrow \langle 1,1,a\rangle$ for $a \geq 3$, the lattice L_0 represents 1 or 2. Consider the case when $1 \longrightarrow L_0$ and the case when $1 \not\longrightarrow L_0$, $2 \longrightarrow L_0$. Using the following properties, we get the desired result.

$$\langle 3,3\rangle \not\longrightarrow \langle 1,1,1,a\rangle \text{ for } a \geq 4,$$

$$\langle 3,5\rangle \not\longrightarrow \langle 1,1,1,1\rangle,\ \langle 3,3\rangle \not\longrightarrow \langle 1,1,1,2\rangle,\ \langle 1,6\rangle \not\longrightarrow \langle 1,1,1,3\rangle,$$

$$\langle 3,3\rangle \not\longrightarrow \langle 1,1,2,a\rangle \text{ for } a \geq 4, \quad \langle 3,3\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & b \end{pmatrix} \text{ for } b \geq 3,$$

$$\langle 3,5\rangle \not\longrightarrow \langle 1,1,2,2\rangle,\ \langle 2,6\rangle \longrightarrow \langle 1,1,2,3\rangle,\ \langle 2,3\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad \square$$

Remark 4.1.2. Every quinary $\mathbb{Z}$-lattices in above theorem is not diagonally 2-universal. We can verify that the following quinary $\mathbb{Z}$-lattices are not diagonally 2-universal.

$$\langle 3,7\rangle \not\longrightarrow \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 7,35\rangle \not\longrightarrow \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix},$$

$$\langle 7,7\rangle \not\longrightarrow \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix},\ \langle 7,7\rangle \not\longrightarrow \langle 1,1,1,3,4\rangle,$$

$$\langle 1,7\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix},\ \langle 1,7\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{pmatrix},$$

$$\langle 1,10\rangle \not\longrightarrow \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},\ \langle 2,6\rangle \not\longrightarrow \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix},$$

$$\langle 3,14\rangle \not\longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 6 \end{pmatrix}.$$

The above theorem and remark say that there are at most 39 diagonally 2-universal quinary $\mathbb{Z}$-lattices. Among those,

$$\langle 1,1,1,1,1\rangle,\ \langle 1,1,1,1,2\rangle,\ \langle 1,1,1,1,3\rangle,\ \langle 1,1,1,2,2\rangle,\ \langle 1,1,1,2,3\rangle,$$

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

are 2-universal (see [8]), and

$$\langle 1,1,1,1,5\rangle,\ \langle 1,1,2,2,3\rangle,\ \langle 1,1,2,2,5\rangle,\ \langle 1,1,2,3,5\rangle,\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

are almost 2-universal $\mathbb{Z}$-lattices whose exceptions are only binary non-diagonal lattices (see [6], [13], and theorems in previous chapter).

4.2 Proof of diagonally 2-universality

In this section, we prove the diagonally 2-universalities of some candidates for diagonally 2-universal quinary $\mathbb{Z}$-lattices. The proof is nearly the same to that in previous chapter.

Theorem 4.2.1. *The following quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,1,1,4\rangle,\ \langle 1,1,2,2,2\rangle,\ \langle 1,1,2,2,4\rangle,$$

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix},\ \textit{and}\ \langle 1,1\rangle \perp \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

are diagonally 2-universal.

Proof. Let L be one of the quinary $\mathbb{Z}$-lattice given above. Then, the class number of L is one and L is 2-universal over $\mathbb{Z}_p$ for any odd prime p. Although L is not 2-universal over $\mathbb{Z}_2$, one can easily check that L_2 represents all binary diagonal $\mathbb{Z}_2$-lattices. Therefore, L represents all binary diagonal $\mathbb{Z}$-lattices. □

For the proof of diagonally 2-universality, we use the same method as previous chapter. We want to show that $\ell \longrightarrow M \perp \langle n\rangle$, where M is a quaternary lattice, n is a positive integer, and ℓ is a binary lattice. In this case, we only consider the binary diagonal lattice ℓ. Let $\ell = \langle a,b\rangle$, and define

$$\ell^n_{s,t} = [a-ns^2, -nst, b-nt^2] \simeq [a-ns^2, nst, b-nt^2].$$

Suppose that s,t are relatively prime to $\mathfrak{P}$, where $\mathfrak{P}$ is the set of primes p such that $\left(\dfrac{dM}{p}\right) = -1$. Then $\mathfrak{s}\ell_{s,t} \not\subseteq p\mathbb{Z}$ for all primes $p \in \mathfrak{P}$. And the remaining process is the same as previous chapter.

Theorem 4.2.2. *The quinary $\mathbb{Z}$-lattice $\langle 1,1,1,3,4\rangle$ represents all binary diagonal lattices except $\langle 7,7\rangle$.*

Proof. Let $\ell = \langle a, b\rangle$ be any binary diagonal lattice. Since the lattices $\langle 1,1,3,4\rangle$, $\langle 1,1,1,4\rangle$ are universal, $\langle 1,b\rangle$, $\langle 3,b\rangle \longrightarrow \langle 1,1,1,3,4\rangle$. Since $\langle 1,2,3,4\rangle$ is universal, $\langle 2,b\rangle \longrightarrow \langle 1,2,2,3,4\rangle \longrightarrow \langle 1,1,1,3,4\rangle$.

Now assume that $a, b > 3$. Consider the quaternary sublattice $M = \langle 1,1,1,4\rangle$. Note that M has class number one and $dM = 2^2$. Thus,

$$\ell \longrightarrow M \text{ if and only if } \ell \longrightarrow M \text{ over } \mathbb{Z}_2.$$

If $\mathrm{ord}_2(ab) = 1$, then $\ell \longrightarrow M$ over $\mathbb{Z}_2$, and hence $\ell \longrightarrow L$. Furthermore, one may easily show that for any $a \equiv b \equiv 3 \pmod 4$, $\langle a, 4b\rangle \longrightarrow M$. If one of a or b, say a, is congruent to 1 modulo 4, then $\mathrm{ord}_2(a-3) = 1$, and hence $\ell^3_{1,0} \longrightarrow M$. Assume that $a \equiv b \equiv 3 \pmod 4$. Suppose that one of a or b, say a, is not contained in $\{7, 19, 23\}$. If $a \equiv 3 \pmod 8$, then $\mathrm{ord}_2(a-3) = 3$ or $a-27$ is positive and $\mathrm{ord}_2(a-27) = 3$. Hence $\ell^3_{1,0} \longrightarrow M$ or $\ell^3_{3,0} \longrightarrow M$. If $a \equiv 7 \pmod 8$, then

$$\frac{a-3}{4} \equiv 3 \pmod 4 \text{ or } a-27 \text{ is positive and } \frac{a-27}{4} \equiv 3 \pmod 4.$$

Therefore, $\ell^3_{1,0} \longrightarrow M$ or $\ell^3_{3,0} \longrightarrow M$. If $a, b \in \{7, 19, 23\}$, then it can be easily shown that all possible binary diagonal lattices except $\langle 7,7\rangle$ are represented by L. If $\mathrm{ord}_2(a) = \mathrm{ord}_2(b) = 1$, then $\ell^3_{1,0} \longrightarrow M$. Finally, if one of a or b is divisible by 4, then ℓ is a sublattice of the binary diagonal lattice considered in the above. Since $\langle 7, 28\rangle \longrightarrow L$, every binary diagonal lattice except $\langle 7,7\rangle$ is represented by L. $\square$

Theorem 4.2.3. *The quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

are diagonally 2-universal. And the quinary $\mathbb{Z}$-lattice

$$\langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

represents all binary diagonal lattices except $\langle 1,7\rangle, \langle 4,7\rangle, \langle 16,7\rangle$.

Proof. Let L be one of the lattices given above. Consider the quaternary lattice $M = \langle 1,1,2,2\rangle$. It has class number one and $dM = 2^2$. Hence, for a binary diagonal lattice ℓ, $\ell \longrightarrow M$ over $\mathbb{Z}$ if and only if $\ell \longrightarrow M$ over $\mathbb{Z}_2$. Note that

$$M \perp \langle 10\rangle \longrightarrow \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad M \perp \langle 12\rangle \longrightarrow \langle 1,1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix},$$

$$M \perp \langle 14\rangle \longrightarrow \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \quad M \perp \langle 18\rangle \longrightarrow \langle 1,1,2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}.$$

Using the similar method as the proof of Theorem 4.2.2, one can show that the lattice $M \perp \langle n\rangle$ represents all binary diagonal lattices with only finitely many exceptions, where $n = 10, 12, 14, 18$. Checking $\ell \longrightarrow L$ for all exceptions, the desired result follows. $\square$

Theorem 4.2.4. *The quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,1,3,3\rangle,\ \langle 1,1,2,3,3\rangle$$

are diagonally 2-universal.

Proof. Note that

$$\big(\langle 1,1,1\rangle \perp [2,1,2]\big)^3 \longrightarrow \langle 1,1,1,3,3\rangle, \quad \big(\langle 1,1,1,2,3\rangle\big)^3 \longrightarrow \langle 1,1,2,3,3\rangle.$$

Since $\langle 1,1,1\rangle \perp [2,1,2]$ and $\langle 1,1,1,2,3\rangle$ are 2-universal (see [8]), given two lattices represents all binary lattices ℓ such that $\mathfrak{s}\ell \subseteq 3\mathbb{Z}$. Now we may only consider the binary diagonal lattices $\ell = \langle a,b\rangle$ such that $(a,b) \not\equiv (0,0) \pmod 3$. Let $M = \langle 1,1,3,3\rangle$. One can easily show that M is 2-universal over $\mathbb{Z}_p$ for any prime greater than 3, and M_2 represents all binary diagonal $\mathbb{Z}_2$-lattices. Therefore, if $\ell \longrightarrow M$ over $\mathbb{Z}_3$, then $\ell \longrightarrow M$ over $\mathbb{Z}$. Assume that ℓ is $\mathbb{Z}_3$-primitive. If $(a,b) \not\equiv (1,2),(2,1) \pmod 3$, then $\ell \longrightarrow M$ over

$\mathbb{Z}_3$. Suppose that $(a, b) \equiv (1, 2) \pmod 3$. Then

$$\langle a, b-1 \rangle \longrightarrow M, \text{ and hence } \langle a, b \rangle \longrightarrow M \perp \langle 1 \rangle,$$

$$\langle a-2, b \rangle \longrightarrow M, \text{ and hence } \langle a, b \rangle \longrightarrow M \perp \langle 2 \rangle, \text{ when } a \geq 4.$$

Since the lattice $\langle 1, 2, 3, 3 \rangle$ is universal by the 15-theorem, the binary lattice $\langle 1, b \rangle$ is represented by $\langle 1, 1, 2, 3, 3 \rangle$ for all b. □

Theorem 4.2.5. *The quinary $\mathbb{Z}$-lattice $L = \langle 1, 1, 1, 3, 5 \rangle$ is diagonally 2-universal.*

Proof. Let $\ell = \langle a, b \rangle$ be any binary diagonal lattice such that $0 < a \leq b$. We always assume that both a and b are square-free integers. By 15-theorem, the quaternary lattices

$$\langle 1, 1, 3, 5 \rangle, \ \langle 1, 2, 3, 5 \rangle, \ \langle 1, 1, 1, 5 \rangle, \ \langle 1, 1, 1, 3 \rangle$$

are universal. Hence $\langle a, b \rangle \longrightarrow L$ whenever $a = 1, 2, 3, 5$. Now we may assume that $a, b \geq 6$.

(Case 1) $(a, b) \not\equiv (0, 0) \pmod 3$ and $(a, b) \not\equiv (0, 0) \pmod 5$
Consider the quaternary sublattice $M = \langle 1, 1, 1, 3 \rangle$ which has class number one. Checking the local structure of the binary lattice $\ell^5_{1,1} = [a-5,\ 5,\ b-5]$, we can verified that $\ell^5_{1,1} \longrightarrow M$ over $\mathbb{Z}_p$ for all primes p. Hence $\ell = \langle a, b \rangle \longrightarrow M \perp \langle 5 \rangle$ if $\ell^5_{1,1}$ is positive definite. For $a, b \geq 6$, there are only finitely many cases that $\ell^5_{1,1}$ is not positive definite. We can check that such lattices ℓ are represented by L.

(Case 2) $(a, b) \equiv (0, 0) \pmod 3$ and $(a, b) \not\equiv (0, 0) \pmod 5$
Consider the lattice $M = \langle 1, 1 \rangle \perp [2, 1, 2]$ which has class number one. From that

$$\left(M \perp \langle 15 \rangle\right)^3 \longrightarrow L,$$

it is enough to show that $\ell = \langle a, b \rangle \longrightarrow M \perp \langle 15 \rangle$ when $3 \nmid ab$ and $(a, b) \not\equiv (0, 0) \pmod 5$. For the binary lattice $\ell^{15}_{1,1} = [a-15,\ 15,\ b-15]$, we cas use the method similar to step 1. Then we get $\ell \longrightarrow M \perp \langle 15 \rangle$ with finitely many exceptions. For example, $\langle 1, 5 \rangle, \langle 7, 7 \rangle$ are not represeted by $M \perp \langle 15 \rangle$.

By a direct calculation, one can show that the binary lattices obtained by scaling these exceptions are indeed represented by L.

(Case 3) $(a,b) \not\equiv (0,0) \pmod 3$ and $(a,b) \equiv (0,0) \pmod 5$
Let $M = \langle 1,1,1,5\rangle$. From that $\left(M \perp \langle 15\rangle\right)^5 \longrightarrow L$, it is enough to show that $\ell = \langle a,b\rangle \longrightarrow M \perp \langle 15\rangle$ when $5 \nmid ab$ and $(a,b) \not\equiv (0,0) \pmod 3$. We can verify the followings:

- If $(a,b) \equiv (1,1) \pmod 4$, then $\ell^{15}_{1,1} \longrightarrow M$;
- If $(a,b) \equiv (1,2),(2,1),(2,3),(3,2),(3,3) \pmod 4$, then $\ell^{15}_{2,2} \longrightarrow M$;
- If $(a,b) \equiv (1,3),(2,2) \pmod 4$, then $\ell^{15}_{1,2} \longrightarrow M$;
- If $(a,b) \equiv (2,2),(3,1) \pmod 4$, then $\ell^{15}_{2,1} \longrightarrow M$.

Using the method similar to step 2, we get the desired result.

(Case 4) $(a,b) \equiv (0,0) \pmod 3$ and $(a,b) \equiv (0,0) \pmod 5$
Let $K = \langle 1,5\rangle \perp [2,1,2]$. From that $\left(\mathrm{gen}(K) \perp \langle 15\rangle\right)^{15} \longrightarrow L$, it is enough to show that $\ell = \langle a,b\rangle \longrightarrow K' \perp \langle 15\rangle$ for some $K' \in \mathrm{gen}(K)$ when $\gcd(ab,15) = 1$. It can be verified that $\ell^{15}_{1,1} \longrightarrow \mathrm{gen}(K)$ for all cases. Using the method similar to step 2, we get the desired result. $\square$

Theorem 4.2.6. *The quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix},\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$$

are diagonally 2-universal.

Proof. Note that

$$\langle 1,1,1,3\rangle \perp \langle 33\rangle,\ \langle 1,1,1,4\rangle \perp \langle 44\rangle,\ \langle 1,1,1,5\rangle \perp \langle 55\rangle$$

are sublattices of $\langle 1,1,1\rangle \perp [3,1,4]$, and

$$\langle 1,1,2,3\rangle \perp \langle 33\rangle,\ \langle 1,1,2,4\rangle \perp \langle 44\rangle$$

are sublattices of $\langle 1,1,2\rangle \perp [3,1,4]$. And all the quaternary lattices

$$\langle 1,1,1,3\rangle, \langle 1,1,1,4\rangle, \langle 1,1,1,5\rangle, \langle 1,1,2,3\rangle, \langle 1,1,2,4\rangle$$

have class number one. Using this, for $a,b \le 33\cdot 3^2 = 297$, we can verify that $\ell = \langle a,b\rangle \longrightarrow L$ where $L = \langle 1,1,1\rangle \perp [3,1,4]$ or $L = \langle 1,1,2\rangle \perp [3,1,4]$.

The remaining process is the same as Theorem 4.2.5. We only provide basic data for the proof.

Let

$$L = \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } K = \langle 1,1,1\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}.$$

Note that K is locally 2-universal, and $\big(\mathrm{gen}(K)\big)^2 \longrightarrow L$. Hence $\langle 2a,2b\rangle \longrightarrow L$ for all a,b. Now we may assume that a or b is odd.

(Case 1) $(a,b) \not\equiv (0,0) \pmod{11}$

Let $M = \langle 1,1,2,3\rangle$, $n = 33$. Note that $M \perp \langle 33\rangle \longrightarrow L$. Then we can verify the followings:

- If $(a,b) \not\equiv (0,0),(0,2),(2,0) \pmod 3$, then $\ell_{1,1} \longrightarrow M$;
- If $(a,b) \equiv (3,6),(6,3),(6,6) \pmod 9$, then $\ell_{1,1} \longrightarrow M$;
- If $(a,b) \equiv (3,3) \pmod 9$, then $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 2 \pmod 3$, $b \equiv 3 \pmod 9$, then $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 3 \pmod 9$, $b \equiv 2 \pmod 3$, then $\ell_{3,1} \longrightarrow M$;
- If $a \equiv 6 \pmod 9$, $b \equiv 2 \pmod 3$, then $\ell_{1,1} \longrightarrow M$ or $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 2 \pmod 3$, $b \equiv 6 \pmod 9$, then $\ell_{1,1} \longrightarrow M$ or $\ell_{3,1} \longrightarrow M$.

(Case 2) $(a,b) \equiv (0,0) \pmod{11}$

Let $K = \langle 1,1,2,11\rangle$. Then $\big(\mathrm{gen}(K) \perp \langle 11\rangle\big)^{11} \longrightarrow L$.

We can verified that $\ell_{1,1}^{11} \longrightarrow \mathrm{gen}(K)$ for all cases.

Let

$$L = \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \ M = \langle 1,1,1,3\rangle \text{ and } n = 33.$$

Note that $M \perp \langle 33\rangle \longrightarrow L$.
For all cases, we can verify the followings:

- If $(a,b) \not\equiv (0,0),(0,2),(2,0) \pmod 3$, then $\ell_{1,1} \longrightarrow M$;
- If $(a,b) \equiv (3,6),(6,3),(6,6) \pmod 9$, then $\ell_{1,1} \longrightarrow M$;
- If $(a,b) \equiv (3,3) \pmod 9$, then $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 2 \pmod 3$, $b \equiv 3 \pmod 9$, then $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 3 \pmod 9$, $b \equiv 2 \pmod 3$, then $\ell_{3,1} \longrightarrow M$;
- If $a \equiv 6 \pmod 9$, $b \equiv 2 \pmod 3$, then $\ell_{1,1} \longrightarrow M$ or $\ell_{1,3} \longrightarrow M$;
- If $a \equiv 2 \pmod 3$, $b \equiv 6 \pmod 9$, then $\ell_{1,1} \longrightarrow M$ or $\ell_{3,1} \longrightarrow M$. □

The followings are data for the proof of the diagonally 2-universality of the five lattices

$$\langle 1,1,2,3,4\rangle,\ \langle 1,1,1,3,6\rangle,\ \langle 1,1,2,3,6\rangle,$$

$$\langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}.$$

By the following lemma, for the proofs of $\langle 1,1,1,3,6\rangle, \langle 1,1,2,3,6\rangle$, we need not consider the case when $(a,b) \equiv (0,0) \pmod 3$.

Lemma 4.2.7. *For any positive integers a,b, the binary diagonal lattice $\langle 3a,3b\rangle$ is represented by quinary $\mathbb{Z}$-lattices*

$$\langle 1,1,1,3,6\rangle,\ \langle 1,1,2,3,6\rangle.$$

Proof. Note that

$$\big(\langle 1,1,2\rangle \perp [2,1,2]\big)^3 \longrightarrow \langle 1,1,1,3,6\rangle,\ \big(\langle 1,1,2,2,3\rangle\big)^3 \longrightarrow \langle 1,1,2,3,6\rangle.$$

Since the lattices $\langle 1,1,2\rangle \perp [2,1,2]$ and $\langle 1,1,2,2,3\rangle$ represent all binary diagonal lattices by [8] and [6], we get the desired result. □

Let

$$L = \langle 1, 1, 1, 3, 6 \rangle$$

and $M = \langle 1, 1, 1, 3 \rangle$, $n = 6$. The followings are the conditions such that $\ell^6_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p = 2, 3$:

(1.1) If $a \equiv 3 \pmod 4$ or $b \equiv 3 \pmod 4$, and $(s,t) \equiv (1,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.2) If $(a,b) \equiv (2,2) \pmod 4$, and $(s,t) \equiv (1,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.3) If $(a,b) \equiv (1,1), (1,2), (2,1) \pmod 4$, and $(s,t) \equiv (1,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(2.1) If $(a,b) \not\equiv (0,0), (0,1), (1,0) \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all s, t;

(2.2) If $a \equiv 3 \pmod 9$, $b \equiv 1 \pmod 3$, and $3 \mid s$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.3) If $a \equiv 6 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 1 \pmod 9$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.4) If $a \equiv 6 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 2 \pmod 9$, $3 \mid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.5) If $a \equiv 15 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 1 \pmod 9$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.6) If $a \equiv 15 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 2 \pmod 9$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.7) If $a \equiv 24 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 1 \pmod 9$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.8) If $a \equiv 24 \pmod{27}$, $b \equiv 1 \pmod 3$, and $s \equiv \pm 1 \pmod 9$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

For all cases, we can verify that $\ell^6_{s,t} \longrightarrow M$ for some

$$(s,t) \in \{(1,1),\ (1,2),\ (1,3),\ (1,6),\ (3,1),\ (3,2)\}.$$

Let

$$L = \langle 1,1,2,3,6 \rangle$$

and $M = \langle 1,1,2,3 \rangle$, $n = 6$. The followings are the conditions such that $\ell^6_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p = 2, 3$:

(1.1) If $(a,b) \equiv (1,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ for all s,t;

(1.2) If $(a,b) \equiv (2,2) \pmod 4$, and $(s,t) \equiv (1,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.3) If $a \equiv 6 \pmod 8$ or $b \equiv 6 \pmod 8$, and $(s,t) \equiv (1,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.4) If $a \nmid 2$, $b \equiv 2 \pmod 8$, and $(s,t) \equiv (0,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.5) If $a \nmid 2$, $b \equiv 2 \pmod 8$, then $\ell_{3,1}, \ell_{3,3} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.6) If $a \equiv 2 \pmod 8$, $2 \nmid b$, and $(s,t) \equiv (1,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(2.1) If $(a,b) \not\equiv (0,0), (0,2), (2,0) \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all s,t;

(2.2) If $a \equiv 3 \pmod 9$, $b \equiv 2 \pmod 3$, and $3 \mid s$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.3) If $a \equiv 6 \pmod{27}$, $b \equiv 2 \pmod 3$, and $s \equiv \pm 1 \pmod 9$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.4) If $a \equiv 6 \pmod{27}$, $b \equiv 2 \pmod 3$, and $s \equiv \pm 2 \pmod 9$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.5) If $a \equiv 15 \pmod{27}$, $b \equiv 2 \pmod{3}$, and $s \equiv \pm 1 \pmod{9}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t;

(2.6) If $a \equiv 15 \pmod{27}$, $b \equiv 2 \pmod{3}$, and $s \equiv \pm 2 \pmod{9}$, $3 \mid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.7) If $a \equiv 24 \pmod{27}$, $b \equiv 2 \pmod{3}$, and $s \equiv \pm 1 \pmod{9}$, $3 \mid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.8) If $a \equiv 24 \pmod{27}$, $b \equiv 2 \pmod{3}$, and $s \equiv \pm 2 \pmod{9}$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.8) If $a \equiv 24 \pmod{27}$, $b \equiv 2 \pmod{3}$, and $s \equiv \pm 4 \pmod{9}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all t.

For all cases, we can verify that $\ell^6_{s,t} \longrightarrow M$ for some

$$(s,t) \in \{(1,1),\ (1,2),\ (1,3),\ (1,6),\ (2,1),\ (2,3),\ (2,5), (3,1),\ (3,2),\ (3,3)\}.$$

Let

$$L = \langle 1, 1, 2, 3, 4 \rangle$$

and $M = \langle 1, 1, 2, 4 \rangle$, $n = 3$. The followings are the conditions such that $\ell^3_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p = 2, 3$:

(1.1) If $(a,b) \equiv (0,1), (1,0) \pmod{2}$ and $(s,t) \equiv (1,1) \pmod{2}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.2) If $(a,b) \equiv (1,1) \pmod{2}$ and $(s,t) \equiv (0,0) \pmod{2}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(2.1) If $(a,b) \not\equiv (0,0) \pmod{3}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$ for all s,t;

(2.2) If $(a,b) \equiv (3,3), (3,6), (6,3) \pmod{9}$ and $(s,t) \equiv (\pm 1, \pm 1) \pmod{3}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.3) If $(a,b) \equiv (6,6) \pmod{9}$ and $(s,t) \equiv (\pm 1, 0) \pmod{3}$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$.

For all cases, we can verify that $\ell^3_{s,t} \longrightarrow M$ for some

$$(s,t) \in \{(1,1),\ (1,3),\ (2,2),\ (2,6)\}.$$

Let

$$L = \langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

and $M = \langle 1,1\rangle \perp [2,1,3]$, $n = 3$. The followings are the conditions such that $\ell^3_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p = 2,3,5$:

(1.1) If $2 \mid a$ and $2 \nmid s$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ for all t;

(1.2) If $2 \nmid a$ and $2 \mid s$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$ for all t;

(2.1) If $3 \nmid a$, $3 \nmid s$ or $3 \nmid b$, $3 \nmid t$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.2) If $(a,b) \equiv (3,3),(3,6),(6,3) \pmod 9$ and $(s,t) \equiv (\pm1,\pm1) \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(2.3) If $(a,b) \equiv (6,6) \pmod 9$ and $(s,t) \equiv (\pm1,0),(0,\pm1) \pmod 3$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_3$;

(3.1) If $5 \nmid ab$, then $\ell_{s_1,t_1} \longrightarrow M$ over $\mathbb{Z}_5$ or $\ell_{s_2,t_2} \longrightarrow M$ over $\mathbb{Z}_5$, where $s_1 = s_2$ and $t_1 \not\equiv \pm t_2 \pmod 5$, or $s_1 \not\equiv \pm s_2 \pmod 5$ and $t_1 = t_2$;

(3.2) If $5 \mid ab$ and $5 \nmid st$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_5$.

For all cases, we can verify that $\ell^3_{s,t} \longrightarrow M$ for some

$$(s,t) \in \{(1,1),\ (1,2),\ (1,3),\ (1,6),\ (2,1),\ (2,2),\ (2,3),\ (2,6)\}.$$

Let

$$L = \langle 1,1,2\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}.$$

(Case 1) $(a,b) \not\equiv (0,0) \pmod 7$
Let $M = \langle 1,2\rangle \perp [3,1,5]$, $n = 1$. The followings are the conditions such that $\ell^1_{s,t} \longrightarrow M$ over $\mathbb{Z}_p$ where $p = 2,7$:

(1.1) If $(a,b) \equiv (1,2),(2,1),(2,3),(3,2) \pmod 4$ and $(s,t) \equiv (0,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.2) If $(a,b) \equiv (2,2) \pmod 4$ and $(s,t) \equiv (0,1),(1,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.3) If $(a,b) \equiv (1,1),(5,5) \pmod 8$, and $(s,t) \equiv (2,0),(0,2) \pmod 4$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.4) If $(a,b) \equiv (1,5),(5,1) \pmod 8$, and $(s,t) \equiv (2,2) \pmod 4$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.5) If $(a,b) \equiv (1,3),(3,3) \pmod 4$, and $(s,t) \equiv (0,1) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(1.6) If $(a,b) \equiv (3,1),(3,3) \pmod 4$, and $(s,t) \equiv (1,0) \pmod 2$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_2$;

(2.1) If $7 \nmid ab$, then $\ell_{s_1,t_1} \longrightarrow M$ over $\mathbb{Z}_7$ or $\ell_{s_2,t_2} \longrightarrow M$ over $\mathbb{Z}_7$, where $s_1 = s_2$ and $t_1 \not\equiv \pm t_2 \pmod 7$;

(2.2) If $7 \mid a$, $7 \nmid b$ or $7 \nmid a$, $7 \mid b$, and $7 \nmid st$, then $\ell_{s,t} \longrightarrow M$ over $\mathbb{Z}_7$.

For all cases, we can verify that $\ell^1_{s,t} \longrightarrow M$ for some

$$(s,t) \in \{(1,2),\ (1,4),\ (2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,6),\ (2,8)\}.$$

(Case 2) $(a,b) \equiv (0,0) \pmod 7$
Let

$$K = \langle 1,1,2,14 \rangle.$$

Then $\big(\mathrm{gen}(K) \perp \langle 7 \rangle\big)^7 \longrightarrow L$. We can verify the followings when $7 \nmid ab$:

(1) If $(a,b) \equiv (1,2),(1,3),(2,1),(2,3),(3,1),(3,2) \pmod 4$, then $\ell_{2,2} \longrightarrow M$;

(2) If $(a,b) \equiv (1,1),(2,2) \pmod 4$, then $\ell_{1,2} \longrightarrow M$;

(3) If $(a,b) \equiv (3,7),(7,3) \pmod 8$, then $\ell_{2,2} \longrightarrow M$;

(4) If $(a,b) \equiv (3,3) \pmod 4$, then $\ell_{2,1} \longrightarrow M$ or $\ell_{2,3} \longrightarrow M$.

To sum up all, we get the following result:

Theorem 4.2.8. *There are 34 diagonally 2-universal quinary $\mathbb{Z}$-lattices found so far, and there can be at most 5 more. They are:*

$$\begin{array}{cccc}
\langle 1,1,1,1,1\rangle, & \langle 1,1,1,1,2\rangle, & \langle 1,1,1,1,3\rangle, & \langle 1,1,1,1,4\rangle, \\
\langle 1,1,1,1,5\rangle, & \langle 1,1,1,2,2\rangle, & \langle 1,1,1,2,3\rangle, & \langle 1,1,1\rangle \perp \begin{pmatrix}2&1\\1&2\end{pmatrix}, \\
\langle 1,1,1\rangle \perp \begin{pmatrix}2&1\\1&3\end{pmatrix}, & \langle 1,1,1,3,3\rangle, & \langle 1,1,1,3,5\rangle, & \langle 1,1,1,3,6\rangle, \\
\langle 1,1,1\rangle \perp \begin{pmatrix}3&1\\1&4\end{pmatrix}, & \langle 1,1,2,2,2\rangle, & \langle 1,1,2,2,3\rangle, & \langle 1,1,2,2,4\rangle, \\
\langle 1,1,2,2,5\rangle, & \langle 1,1,2\rangle \perp \begin{pmatrix}2&1\\1&2\end{pmatrix}, & \langle 1,1,2\rangle \perp \begin{pmatrix}2&1\\1&3\end{pmatrix}, & \langle 1,1,2\rangle \perp \begin{pmatrix}2&1\\1&4\end{pmatrix}, \\
\langle 1,1,2\rangle \perp \begin{pmatrix}2&1\\1&5\end{pmatrix}, & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&2&1\\1&1&2\end{pmatrix}, & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&2&1\\1&1&3\end{pmatrix}, & \langle 1,1\rangle \perp \begin{pmatrix}2&1&0\\1&2&1\\0&1&3\end{pmatrix}, \\
\langle 1,1\rangle \perp \begin{pmatrix}2&1\\1&2\end{pmatrix} \perp \langle 3\rangle, & \langle 1,1,2,3,3\rangle, & \langle 1,1,2,3,4\rangle, & \langle 1,1,2,3,5\rangle, \\
\langle 1,1,2,3,6\rangle, & \langle 1,1,2\rangle \perp \begin{pmatrix}3&1\\1&3\end{pmatrix}, & \langle 1,1,2\rangle \perp \begin{pmatrix}3&1\\1&4\end{pmatrix}, & \langle 1,1,2\rangle \perp \begin{pmatrix}3&1\\1&5\end{pmatrix}, \\
\langle 1,1,2\rangle \perp \begin{pmatrix}3&1\\1&6\end{pmatrix}^{\sharp} & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&0\\1&0&3\end{pmatrix}, & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&0\\1&0&6\end{pmatrix}^{\sharp}, & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&3\end{pmatrix}^{\sharp}, \\
\langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&4\end{pmatrix}^{\sharp}, & \langle 1,1\rangle \perp \begin{pmatrix}2&0&1\\0&3&1\\1&1&5\end{pmatrix}^{\sharp}, & \langle 1,1\rangle \perp \begin{pmatrix}2&1&1\\1&3&0\\1&0&3\end{pmatrix} &
\end{array}$$

The $\mathbb{Z}$-lattices with $\sharp$-mark are not yet determined to be diagonally 2-universal.

4.3 Criterion for diagonally 2-universality

For a $\mathbb{Z}$-lattice L, it is diagonally 2-universal if it contains a diagonally 2-universal quinary sublattice. But the converse is not true. Consider the $\mathbb{Z}$-lattice

$$L = \langle 1,1,1\rangle \perp \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix},$$

which does not contain any diagonally 2-universal quinary sublattices. Note that L represents the lattices $\langle 1,1,1,3,4\rangle$ and $\langle 7,7\rangle$. Hence L is diagonally 2-universal by Theorem 4.2.2. For the characterization of diagonally 2-universality, we need information about all quinary $\mathbb{Z}$-lattices in Theorem 4.1.1, as well as diagonally 2-universal quinary $\mathbb{Z}$-lattices.

By [9], there exists a finite set S consisting of binary diagonal $\mathbb{Z}$-lattices such that any $\mathbb{Z}$-lattice representing every element of S is diagonally 2-universal. The following theorem gives the partial answer.

Theorem 4.3.1. *A diagonal $\mathbb{Z}$-lattice is diagonally 2-universal if and only if it represents the following seven binary diagonal $\mathbb{Z}$-lattices in the set:*

$$\mathfrak{D} = \{\langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 1,6\rangle,\ \langle 2,6\rangle,\ \langle 3,3\rangle,\ \langle 3,5\rangle,\ \langle 7,7\rangle\}.$$

Moreover, a set $\mathfrak{D}'$ of binary diagonal $\mathbb{Z}$-lattices satisfies the condition that every diagonal $\mathbb{Z}$-lattice representing all binary diagonal lattices in $\mathfrak{D}'$ is diagonally 2-universal if and only if $\mathfrak{D}'$ contains

$$\mathfrak{D}_{\alpha,\beta} = \{\langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle \alpha,6\rangle,\ \langle \beta,6\rangle,\ \langle 3,3\rangle,\ \langle 3,5\rangle,\ \langle 7,7\rangle\},$$

where $\alpha = 1$ or 4 and $\beta = 2$ or 5.

Proof. Let L be a diagonal $\mathbb{Z}$-lattice which represents all binary diagonal lattices in $\mathfrak{D}$. From the proof of Theorem 4.1.1, we see that any diagonal lattice representing the following six binary diagonal lattices,

$$\langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 1,6\rangle,\ \langle 2,6\rangle,\ \langle 3,3\rangle,\ \text{and } \langle 3,5\rangle,$$

has a sublattice which is isometric to one of the quinary diagonal lattices in the list of Theorem 4.1.1. These quinary diagonal lattices are all diagonally 2-universal except the lattice $\langle 1,1,1,3,4\rangle$. Since $\langle 7,7\rangle \longrightarrow L$, for all cases, L is diagonally 2-universal by Theorem 4.2.2.

For the second assertion, let $\mathfrak{D}'$ be a set of binary diagonal lattices such that every diagonal lattice that represents all binary diagonal lattices in $\mathfrak{D}'$ is diagonally 2-universal. Since the lattice $\langle 1,1,1,1,1\rangle$ is 2-universal, the diagonal lattice $\langle 2,2,2,2,2\rangle \perp \langle 1,3\rangle$ represents all binary diagonal lattices except $\langle 1,1\rangle$. By [6], the diagonal lattice $\langle 1,1,1,2,4\rangle$ represents all binary lattices except $\langle 3,3\rangle$. The diagonal lattice $\langle 1,1,1,3,4\rangle$ represents all binary diagonal lattices except $\langle 7,7\rangle$ by Theorem 4.2.2. One may show that the diagonal lattice $\langle 1,1,1,1,6\rangle$ and $\langle 1,1,3,3,4,5\rangle$ represent all binary diagonal lattices except $\langle 3,5\rangle$ and $\langle 1,2\rangle$, respectively. Hence, $\mathfrak{D}'$ should contain the five binary diagonal lattices

$$\langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 3,3\rangle,\ \langle 3,5\rangle \text{ and } \langle 7,7\rangle.$$

The diagonal lattice $\langle 1,1,1,3,7,8\rangle$ represents all binary diagonal lattices except $\langle 1,6\rangle$ and $\langle 4,6\rangle$ by Theorem 3.1.2. Hence, $\mathfrak{D}'$ should contain the binary quadratic form $\langle 1,6\rangle$ or $\langle 4,6\rangle$. The diagonal lattice $\langle 1,1,2,3,7,8\rangle$ represents all binary diagonal lattice except $\langle 2,6\rangle$ and $\langle 5,6\rangle$ by Theorem 3.1.3. Therefore, $\mathfrak{D}'$ should contain $\mathfrak{D}_{\alpha,\beta}$, where $\alpha = 1$ or 4 and $\beta = 2$ or 5.

Conversely, one may easily show that any diagonal lattice representing all binary diagonal lattices in $\mathfrak{D}_{\alpha,\beta}$ is diagonally 2-universal for any α and β given above. In fact, the assertion follows directly from Theorem 4.1.1 if we use the fact that $\langle 1,6\rangle \simeq \langle 4,6\rangle$ and $\langle 2,6\rangle \simeq \langle 5,6\rangle$ over $\mathbb{Z}_3$. This completes the proof. □

Remark 4.3.2. Note that there exist *non-diagonal* $\mathbb{Z}$-lattices that represent all binary diagonal lattices contained in the set $\mathfrak{D}$, but are not diagonally 2-universal. By Theorem 3.2.2 and 3.2.1, the quinary $\mathbb{Z}$-lattices

$$\langle 1,1,3\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } \langle 1,1,5\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

represent all binary diagonal lattices except only one, and the exceptions are $\langle 1, 10\rangle$ and $\langle 2, 3\rangle$, respectively. By Theorem 4.2.3, the quinary $\mathbb{Z}$-lattice

$$\langle 1, 1\rangle \perp \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

represents all binary diagonal lattices except $\langle 1, 7\rangle, \langle 4, 7\rangle, \langle 16, 7\rangle$. From the fact that the quinary $\mathbb{Z}$-lattice

$$\langle 1, 1, 1\rangle \perp \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

has class number one, we can verify that it represents all binary lattices ℓ such that $\ell \not\cong \langle 3, 7\rangle$ over $\mathbb{Z}_2$. These four $\mathbb{Z}$-lattices represent all binary diagonal lattices in the set $\mathfrak{D}$, but they are not diagonally 2-universal.

We believe that a $\mathbb{Z}$-lattice is diagonally 2-universal if and only if it represents all binary diagonal lattices in the set

$$\mathfrak{D} \cup \{\langle 1, 7\rangle,\ \langle 1, 10\rangle,\ \langle 2, 3\rangle,\ \langle 3, 7\rangle,\ \langle 3, 14\rangle,\ \langle 7, 35\rangle\}.$$

Bibliography

[1] M. Bhargava, *On the Conway-Schneeberger fifteen theorem*, Contem. Math. **272**(2000), 27-38.

[2] M. Bhargava and J. Hanke, *Universal quadratic forms and the 290-theorem*, Invent. Math., to appear.

[3] J. Bochnak and B.-K. Oh, *Almost universal quadratic forms: an effective solution of a problem of Ramanujan*, Duke Math. J. **147** (2009), no. 1, 131-156.

[4] P. R. Halmos, *Note on almost-universal forms*, Bull. Amer. Math. Soc. **44**(1938), 141-144.

[5] J. S. Hsia, and M. Jöchner, *Almost strong approximations for definite quadratic spaces*, Invent. Math. **129**(1997), 471-487.

[6] D.-S. Hwang, *On almost* 2*-universal integral quinary quadratic forms*, Ph. D. Thesis, Seoul National Univ. (1997).

[7] Y.-S. Ji, M. J. Kim and B.-K. Oh, *Diagonal quadratic forms representing all binary diagonal quadratic forms*, Ramanujan J., to appear.

[8] B. M. Kim, M.-H. Kim and B.-K. Oh, 2*-universal positive definite integral quinary quadratic forms*, Contem. Math. **249**(1999), 51-62.

[9] B. M. Kim, M.-H. Kim and B.-K. Oh, *A finiteness theorem for representability of quadratic forms by forms*, J. Reine Angew. Math. **581** (2005), 23-30.

[10] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge University Press, Cambridge (1993).

[11] H. D. Kloosterman, *On the representation of numbers in the form* $ax^2 + by^2 + cz^2 + dt^2$, Acta Math. **49**(1926), 407-464.

[12] B.-K. Oh, *On universal forms*, Ph. D. Thesis, Seoul National Univ. (1999).

[13] B.-K. Oh, *The representation of quadratic forms by almost universal forms of higher rank*, Math. Z. **244**(2003), 399-413.

[14] B.-K. Oh, *Universal Z-lattices of minimal rank*, Proc. Amer. Math. Soc. **128**(2000), 683–689.

[15] O. T. O'Meara, *The integral representations of quadratic forms over local fields*, Amer. J. Math. **80**(1958), 843-878.

[16] O. T. O'Meara, *Introduction to quadratic forms*, Springer, New York (1963).

[17] G. Pall, *The completion of a problem of Kloosterman*, Amer. J. Math. **68**(1946), 47-58.

[18] G. Pall and A. Ross, *An extension of a problem of Kloosterman*, Amer. J. Math. **68**(1946), 59-65.

[19] S. Ramanujan, *On the expression of a number in the form* $ax^2+by^2+cz^2+dw^2$, Proc. Cambridge Phil. Soc. **19**(1917), 11-21.

[20] M. F. Willerding, *Determination of all classes of (positive) quaternary quadratic forms which represent all positive integers*, Bull. Amer. Math. Soc. **54**(1948), 334-337.

국문초록

이 논문에서는 5변수 이차형식에 의한 2변수 이차형식의 표현에 관련된 다양한 문제를 다룬다. (양의 부호, 정수 계수) 이차형식 중에서 유한 개를 제외한 모든 (양의 부호, 정수 계수) 2변수 이차형식을 표현하는 것을 '거의 모든 2-보편'이라고 한다. 거의 모든 2-보편인 5변수 대각 이차형식의 후보 3개가 실제로 거의 모든 2-보편이 된다는 것을 증명한다. 그리고 예외가 하나뿐인 거의 모든 2-보편인 5변수 이차형식이 최대 16개가 있고, 이 후보 중 몇 개의 거의 모든 2-보편성을 증명한다. (양의 부호, 정수 계수) 이차형식 중에서 (양의 부호, 정수 계수) 2변수 대각 이차형식을 모두 표현하는 것을 '대각 2-보편'이라고 한다. 대각 2-보편인 5변수 이차형식이 최대 34개가 있고, 이 후보 중에서 29개의 대각 2-보편성을 증명한다.

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